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A UNIFIED THEORY OF CONJUGATE FLOWS

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CONTENTS

Various examples of flow systems are known in which the study of conjugate flows (i.e. flows uniform in the direction of streaming which separately satisfy the hydrodynamical equations) is crucial to the understanding of observed wave phenomena. Open-channel flows are the best-known example, with which remarkable qualitative similarities have been revealed in studies of other systems: for instance, it has appeared in general that any pair of conjugate flows is *transcritical* (i.e. if one flow is supercritical according to a generalized definition, then the other is subcritical). So far the common ground among theoretical treatments has been defined only by intuitive analogies, and the aim of this paper is to give unity to the whole subject by identifying the elements that are intrinsically responsible for universal properties. The problem is accordingly considered in the form of an abstract (nonlinear) operator equation, whose solu-

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tion representing a conjugate flow is a vector in a linear space of finite or infinite dimensions: all known examples are reducible to this form and other applications may be anticipated. The generalized treatment on these lines must have recourse to new methods, however, of a more powerful kind than would suffice for the *ad hoc* treatment of particular examples. A résumé of the required mathematical material is presented in §2.

The main substance of the paper is in §3. In §3.1 the supercritical and subcritical classification of flows is explained generally, being shown to depend on the eigenvalues of the Fréchet derivative of the nonlinear operator presented by the hydrodynamical problem. In §3.3 fixed-point principles are used to define general conditions under which the existence of conjugate flows in a proposed category is guaranteed, and also in this subsection a special argument is given to exemplify the transcritical property of conjugate flows. Several aspects are covered in §3.4 by means of index theory, in particular the problem of classifying a multiplicity of conjugate flows possible in a given system and the question of what conditions ensure uniqueness. In §3.5 variational methods are used to account for the differences in flow force that appear to be an essential attribute of frictionless conjugate flows (flow force is a scalar property which is generally stationary to small variations about a solution of the hydrodynamical equations). The last three sections of the paper present treatments of specific examples illustrating the unified viewpoint given by the theory.

Proofs of two topological theorems used in §3.4 are presented in appendix 1, and in appendix 2 the reasons for the variational significance of flow force are examined.

1. Introduction

In theories of steady waves in fluid media that are both dispersive and nonlinear, the problem is often reducible to consideration of an equation, or system of equations, in the form

$$\phi = \mathscr{A}(\mu)\,\phi, \tag{1.1}$$

where $\mathscr{A}(\mu)$ is a nonlinear operator depending on a positive parameter μ which determines the wave velocity. Equivalently, if axes are taken moving in step with progressive waves, or if the waves are considered in the first place as a stationary phenomenon occurring in a flow, then μ determines the steady-flow velocity in the direction of the principal axis x. The dependent variable ϕ may represent, for instance, displacements or perturbations of a stream-function, and the equation has the trivial solution $\phi \equiv 0$ corresponding to the primary, undisturbed state upon which the waves are supposed to arise. In a number of important cases such an equation is found to have at least one other solution independent of x, which therefore represents a waveless flow distinct from the primary one. Thus there exists a non-trivial solution of the equation, say

$$\phi = \boldsymbol{A}(\mu)\,\phi, \tag{1.2}$$

to which (1.1) reduces when x-dependence is neglected. The pair of flows represented by the null solution and the second solution of (1.2) are called *conjugate*, and their properties are the subject of this paper. Conjugate flows are capable of experimental realization, in a way that will be explained presently, and an understanding of them is generally vital to the interpretation of wave phenomena described by (1.1).

[An example serving to put these ideas into better focus is provided by the flow of continuously stratified heavy fluid between horizontal planes $y = 0$ and $y = 1$ (cf. Ter-Krikorov 1963; also §6 and appendix 2 below). The perturbation $\phi(x, y)$ of a modified stream-function is found to satisfy an elliptic equation

$$\Delta\phi + F(y, \phi; \mu) = 0,$$

in which $F(y, \phi; \mu)$ is a nonlinear function of ϕ such that $F(y, 0; \mu) = 0$. The boundary conditions are $\phi(x, 0) = 0$, $\phi(x, 1) = 0$ and, if periodic waves are in question, a periodicity condition with respect to a given interval of the horizontal coordinate x. Hence the problem is equivalent to solving

$$\phi(x, y) = \mathscr{B}F\{y, \phi(x, y); \mu\}, \tag{1.3}$$

where $\mathscr{B}$ is the linear integral operator whose kernel is the (positive) Green function for $-\Delta$ and the specified boundary conditions. On the assumption that ϕ is independent of x, this equation reduces to

$$\phi(y) = \boldsymbol{B}F\{y, \phi(y); \mu\}, \tag{1.4}$$

where $\boldsymbol{B}$ is the integral operator whose kernel is the one-dimensional Green function for $-\mathrm{d}^2/\mathrm{d}y^2$ and the zero conditions at $y = 0$ and $y = 1$ (see equation (6.18)).]

The notion of conjugate flows is most familiar from the study of streams of water in open channels. In classifying the possible steady frictionless flows along a uniform horizontal channel, Benjamin & Lighthill (1954) pointed out that three physical constants can be used as parameters: the volume flux Q; the total head, or stagnation pressure, R; and the 'flow force' S, defined as the sum of horizontal momentum and pressure force. It appears that with two of these parameters fixed, a certain interval within which the value of the remaining parameter may lie corresponds to a continuous spectrum of periodic waves. For uniform flows, on the other hand, fixed values of two of the parameters allow at most two values of the other, and the two possible flows thus represented are conjugate in the general sense understood here. When the two admissible values of the free parameter are nearly the same, they close the spectrum of periodic waves which are long, approximately 'cnoidal' waves in this case; but otherwise the spectrum may fill only part of the open interval between conjugate flows, the remainder being excluded by wave breaking. In the elementary theory of hydraulic jumps (Lamb 1932, p. 280), Q and S are assumed to be conserved through a steady transition between uniform flows, and it is concluded that the conjugate flow with greater depth must occur on the downstream side because this flow has smaller R, so that a loss rather than a gain of energy takes place at the transition. This model of a dissipative but flow-force conserving transition has no simple counterpart in other systems, and a more effective rationale for various phenomena, particularly for weak, undular jumps, is given by consideration of conjugate flows for which Q and R are the same and S has different values. It is to such energy conserving conjugate flows that the ideas of this paper are most naturally applicable.† In other systems as in the open-channel example, a transition between two such flows can be realized when an obstacle is fixed in the stream, for then the difference in S is balanced by the external force holding the obstacle in place.

A well-known fact about open-channel flows is that one in any pair of conjugates is *supercritical* and the other is *subcritical*. These terms may be defined in several ways which turn out to be largely equivalent, and the definition adopted in this paper is the one that bears most closely on the general mathematical problem expressed by (1.2). First, a flow is called *critical* if an infinitesimal wave of extreme length can be superposed on it: analytically this condition means that the equation determining possible uniform flows has coincident solutions or, what amounts to the same thing, that the derivative of the equation also has a solution. Then a supercritical state of flow is defined as such that, with other features fixed, the x-component of velocity would have to be scaled down in order to produce a critical state; and conversely the velocity would have to be

† An interesting example of a quite different kind is given by a stream of viscous liquid flowing under the action of gravity down an inclined channel. A change in the volume flux from Q_1 to Q_2 at entry to the channel will result in a new régime becoming established whose front propagates downstream at the kinematic wave velocity $(Q_1-Q_2)/[\Sigma(Q_1)-\Sigma(Q_2)]$, where $\Sigma(Q)$ expresses the cross-sectional area of steady flows as a function of Q. From a frame of reference moving with the wave front, there appears a transition between two flows that are respectively supercritical and subcritical with respect to infinitesimal kinematic waves. Total head and flow force have no meaning in this example, of course, because the fluid is viscous; but certain constants for uniform flows of this type may be considered to complete a formal analogy with conjugate open-channel flows. If the density or viscosity of the fluid is stratified, a more complicated problem is posed but it seems that present ideas are still broadly applicable.

scaled up to change a subcritical state into critical. Thus, as the parameter μ in (1.2) is taken to depend inversely on the velocity scale, the flow is called supercritical or subcritical according as $\mu < \mu_c$ or $\mu > \mu_c$, where μ_c is the critical value. If, in a sense to be made precise later, the operator $A(\mu)$ and its derivative are increasing with μ, a useful interpretation of the terms supercritical and subcritical is that the derivative operator is respectively too weak or too strong for there to be another solution in the neighbourhood of the solution in question. A common implication concerning the original problem expressed by (1.1) is that a solution in the form of an infinitesimal sinusoidal wave exists if the primary flow is subcritical, whereas none is possible if the flow is supercritical.† These are often taken as defining properties (see, for example, Benjamin 1962, 1966), but they are tied to special attributes of the operator $\mathscr{A}$ (cf. appendix 2) and so the present definition of supercritical and subcritical is more fundamental in the context of the general conjugate-flow problem.

The transcritical property of conjugate flows has been discovered in studies of other flow systems, and accordingly analogies with open-channel flows have appeared helpful. Another remarkable property of open-channel flows is that the subcritical member of an energy conserving conjugate pair has the greater flow force S, and this too has been found to hold generally among analogous systems, although its demonstration has been a matter of considerable complexity. These ideas have been applied to the explanation of phenomena in vortex flows by Benjamin (1962, 1965, 1967), and the relevant theory has been developed further by Fraenkel (1967) and Sheer (1968). Applications to flows of density-stratified fluids have been discussed by Benjamin (1966) and others, and a recent paper by Van Wijngaarden (1968) has indicated another field of application to flows of liquid–gas mixtures.‡ Various interpretations propounded in this previous work rest heavily on physical intuition, however, and the lack of analytical correspondence between the subjects of analogies is obviously unsatisfactory from a theoretical standpoint. The present aim, accordingly, is to develop a unified theory of conjugate flows by studying the abstract equation (1.2), to whose form the hydrodynamical equations are reducible in all examples so far investigated and for which other applications may be anticipated. Abstract methods will be used in another paper (Benjamin 1971) dealing with conjugate vortex flows and associated wave phenomena, the relation between the two topics being investigated further in this particular application than can suitably be attempted here.

The gist of the general theory may be indicated at once by noting that the various arguments to be used all reduce to obvious considerations in the simplest case, exemplified by open-channel flows (§ 4 below), where ϕ is just a scalar variable. Then the equation determining possible flows has the form

$$\phi = A(\phi; \mu), \tag{1.5}$$

† The essentials of this aspect are these. Linearizing (1.1) about its null solution and supposing that the linearized equation has a solution in the form $\xi(x, y) = \hat{\xi}(y) \sin kx$, where y is the coordinate (or system of coordinates) for the flow cross-section, one generally obtains an x-independent equation

$$\xi = \boldsymbol{C}(k^2, \mu)\, \xi,$$

which is comparable with the linearized version of (1.2). If the effects of the parameters k^2 and μ on the linear operator $\boldsymbol{C}$ are opposing, then a solution with $k^2 > 0$ (i.e. representing a periodic infinitesimal wave) will require $\mu > \mu_c$, where the critical value μ_c is that for which (1.2) has an infinitesimal solution. The same considerations can be made in respect of linearization about a nontrivial solution of (1.2) representing a conjugate flow.

‡ In his study the gas is assumed to be distributed through the liquid in small bubbles of equal mass content, whose concentration is uniform over each cross-section of the flow. Thus the theoretical model is one-dimensional, and for moderately long waves of small amplitude it simulates dispersive and nonlinear effects analogous to those in open-channel flows. Present ideas would still have bearing if these restrictive assumptions were relaxed, although the formal analogy with long water waves would then disappear.

where the right-hand side is a real function of ϕ in the ordinary sense. To emphasize the correspondence with other cases covered by the theory, we may describe $A(z;\mu)$, defined for all real numbers z, as a mapping of the one-dimensional Euclidean space R_1 into itself. The physical problem will generally provide that $A(z;\mu)$ is a continuous function of both variables in a certain range, and as was explained earlier it is so formulated that $A(0;\mu) = 0$. Now suppose, for example, that this function is continuous for all positive values of the arguments, also that

$$|z - A(z,\mu)| > 0 \quad \text{if} \quad z < 0, \tag{1.6}$$

and

$$A(z,\mu) \geqslant 0 \quad \text{if} \quad z \geqslant 0. \tag{1.7}$$

The condition (1.6) makes a negative solution of (1.5) impossible, and so attention focuses on the properties of A on the set of non-negative numbers, which (1.7) shows to be mapped into itself by A. From what has been said earlier it is apparent that the value of the right-hand derivative at the origin,

$$A'(0^+;\mu) = \lambda_0, \quad \text{say,}$$

determines whether the primary flow is supercritical or subcritical with respect to the same 'mode' as a possible conjugate flow (i.e. with respect to infinitesimal long-wave disturbances for which the variable ϕ takes only non-negative values). Let $\lambda_0 = 1$ when $\mu = \mu_c$. Moreover, suppose that μ depends inversely on the velocity scale of the flow, and that

$$A'(z,a) > A'(z;b) > 0 \quad \text{if} \quad a > b > 0,\ z > 0.$$

Then clearly the primary flow is supercritical, critical or subcritical accordingly as $\mu < \mu_c$, $\mu = \mu_c$ or $\mu > \mu_c$. Similarly, the state relative to critical of a conjugate flow represented by a non-trivial solution of (1.5) depends on whether

$$A'(\phi;\mu) = \lambda_\phi, \quad \text{say,}$$

is greater or less than 1.

Various conditions can be stated that will ensure the existence of a positive solution of (1.5). For instance, let us assume that A has an asymptotic derivative,

$$A'(\infty;\mu) = \lambda_\infty, \quad \text{say,}$$

which by virtue of the prior assumption (1.7) cannot be negative. Then, together with the continuity of A (which is obviously essential), either of the pairs of conditions

$$\lambda_0 > 1, \quad \lambda_\infty < 1, \tag{1.8}$$

or

$$\lambda_0 < 1, \quad \lambda_\infty > 1 \tag{1.9}$$

is evidently sufficient for the existence of a point $\phi > 0$ satisfying (1.5). Possibilities allowed by the conditions (1.8) are illustrated in figure 1. It is clear that uniqueness of the non-trivial solution ϕ cannot be guaranteed without some additional assumption. But suppose, for instance, that in keeping with (1.8) the nonlinear function A satisfies

$$A(tz;\mu) > tA(z;\mu) \quad \text{if} \quad z > 0,\ 0 < t < 1. \tag{1.10}$$

The conditions (1.8) and (1.10) evidently imply a unique positive solution ϕ, and moreover that $\lambda_\phi < 1$. Thus there is a unique supercritical flow conjugate to the subcritical primary flow. This case is illustrated in figure 1 (*a*).

Examples in which the solution may not be unique still bear out the transcritical property of conjugate flows in obvious respects. For illustration, let us again take the primary flow to be subcritical and assume the second condition in (1.8) also to be satisfied. Then, if the case $\lambda_\phi = 1$ is excluded as a possibility, the following facts become immediately evident on consideration of the possible graph of the function A, as in figure 1 (*b*): The number of positive solutions must be odd, say $1+2N$, of which $1+N$ represent supercritical conjugate flows ($\lambda_\phi < 1$) and N represent subcritical conjugate flows ($\lambda_\phi > 1$). The flow represented by the least ϕ, which may be called

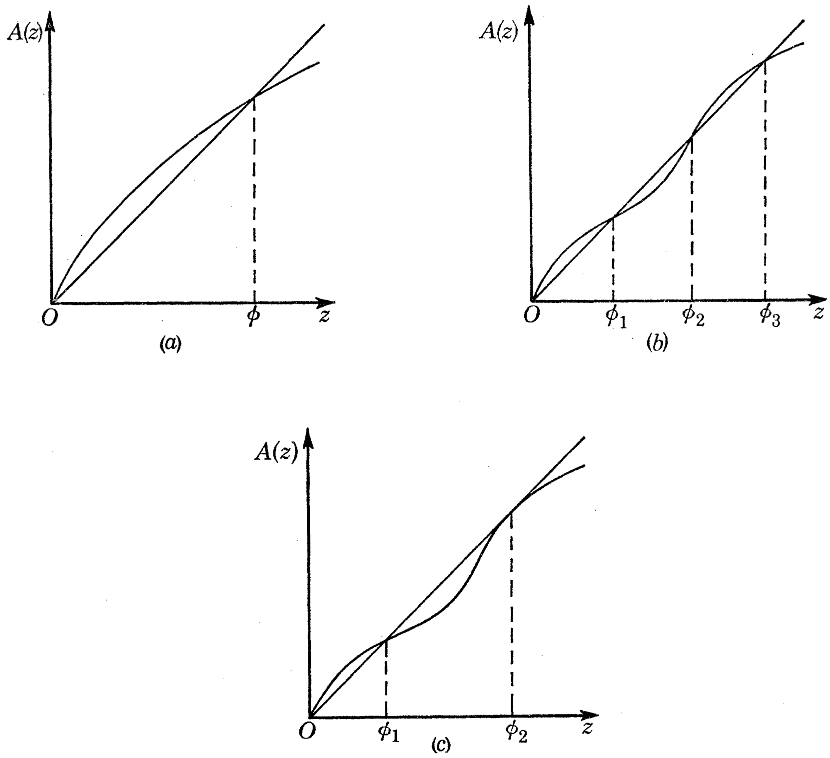

FIGURE 1. Positive solutions of equation (1.5).

the principal conjugate respective to the given primary flow, is always supercritical. Taken in order of size, the remaining $2N$ solutions form N transcritical pairs. Corresponding statements, with the words supercritical and subcritical interchanged, can be made if instead of (1.8) the conditions (1.9) are satisfied. A qualification referring to the exceptional possibility $\lambda_\phi = 1$ is clearly necessary to a general account of even this simplest example, and so it is to be expected that the possibility of precisely critical conjugate flows will obtrude as a slightly awkward aspect of the abstract theory [cf. Fraenkel's (1967) discussion of conjugate vortex flows]. This possibility appears physically extraordinary, however, and in cases such as illustrated in figure 1 (*c*) the solution is unstable in the sense that the slightest change in the specifications of the problem may make it disappear.

Counterparts of the foregoing conclusions will be established for the general class of problems implied by equation (1.2), whose solution ϕ can always be regarded as a vector in some appropriate linear space with finite or infinite number of dimensions. To illustrate this viewpoint, consider the example of a composite fluid which is stably stratified in $n+1$ discrete layers and bounded by horizontal planes. There are n free interfaces, and so the solution giving the vertical displacements of these relative to their levels in the primary flow can be considered as an n-component column vector. Equation (1.2) then has a matrix form, representing a set of simultaneous equations

$$\phi_i = A_i(\phi_1, \phi_2, \ldots, \phi_n; \mu) \quad (i = 1, 2, \ldots, n), \tag{1.11}$$

which involves n distinct nonlinear functions A_i (see § 5 for a specific example with $n = 2$). Correspondingly, when a flow system in question has an infinite number of degrees of freedom, as in the example of a continuously stratified fluid, the solution may be considered as a vector in an infinite-dimensional Banach space. The theory then depends on the assumption that the nonlinear operator $\boldsymbol{A}$ is completely continuous (or compact), which extends the continuity requirement that is essential in the finite-dimensional case—as was pointed out above regarding the case of just one dimension. In the treatment of the abstract problem topological methods will largely be used, in particular concerning the geometrical properties of cones in Banach spaces. Fixed-point principles will be used to establish conditions under which equation (1.2) has a nontrivial solution of a proposed kind; and with regard to the supercritical and subcritical characterization of flows, it will be found helpful to introduce the ideas of, first, the Fréchet derivative of the operator $\boldsymbol{A}$ and, secondly, the Leray–Schauder index of a fixed point. Finally, an account of the essential flow-force differences between conjugate flows will be developed by means of a variational argument. When a new dependent variable ζ is introduced and the governing equation is recast in the form $\zeta = \boldsymbol{G}\zeta$, where the new operator $\boldsymbol{G}$ is such that the vector $u - \boldsymbol{G}u$ is the gradient of a functional $\Lambda(u)$ defined on a Hilbert space (i.e. of a scalar function of position in the space), then the solution ζ is a stationary point of Λ and it generally turns out that $\Lambda(\zeta)$ gives the relative value of flow force for the respective conjugate flow. (The reason for the latter property will be discussed in appendix 2.) A similar argument was applied to the study of conjugate vortex flows by Benjamin (1962), relying on methods of the classical calculus of variations; but Hilbert-space concepts appear to be needed to make a general approach of this kind exact.

The required mathematical propositions are presented in § 2. Accounts of the theories from which this material is taken may be found particularly in three works by Krasnosel'skii: his review article on problems of nonlinear analysis (1958), his monograph on topological methods applied to the study of nonlinear integral equations (1964), and his monograph on positive solutions of operator equations (1964). Since frequent reference is made to these works, the abbreviations K (a), K (b) and K (c) are used for them respectively. In § 3 applications to the generalized hydrodynamical problem are considered and a theory of conjugate flows is derived. The last three sections of the paper deal with examples of different flow systems which illustrate the unified viewpoint given by the theory.

It must be recognized that these examples are amenable to *ad hoc* treatment by means considerably simpler than the abstract theory, and our main object is not to develop methods for solving particular problems. Rather, it is to demonstrate the mathematical unity of an interesting range of problems in fluid dynamics whose treatments by traditional analytical methods appear superficially to have little in common. In this way a rational foundation is given to the intuitive analogies that have hitherto been drawn, regarding wave phenomena seemingly related to familiar open-channel effects.

To anticipate extensions of the present work, it is noted that most ideas of the abstract conjugate-flow theory are also applicable to the more general problem represented by equation

(1.1). Thus solutions describing waves may be proved to exist and classified in much the same way as the x-independent solutions considered here. The use of index theory, as exemplified in § 3.4, appears particularly promising in this regard. A few facts concerning the more general problem are presented in appendix 2, being required to elucidate aspects of the conjugate-flow theory.

2. Mathematical résumé

This section of the paper summarizes the concepts and results from functional analysis that will later be used in the discussion of the conjugate-flow problem. It is intended particularly for readers without much familiarity with functional analysis, and aims at being sufficiently detailed to enable them *ab initio* to understand the gist of all the subsequent arguments.

2.1. *Definitions*

We take E to denote a real Banach space. The norm of an element $u \in E$ is written $\|u\|$, and the zero element of E is denoted by θ. We shall refer to a particular element as a point in the space, or as a vector having the magnitude $\|u\|$ and the direction of the ray from θ to the point u. Examples in view include: the space C of continuous functions $u(y)$ defined on the closure $\bar{D}$ of a bounded domain D in a Euclidean space R_n (e.g. the interval [0, 1] when y is a scalar), for which the norm is

$$\|u\| = \max_{y \in \bar{D}} |u(y)|$$

(Liusternik & Sobolev 1961, pp. 10 and 18); the spaces L_p ($p \geqslant 1$) of (equivalence classes of) functions that are pth power summable on a bounded domain D, for which the norms are

$$\|u\| = \left(\int_D |u(y)|^p \, dy\right)^{\frac{1}{p}};$$

in particular the Hilbert space L_2 (Liusternik & Sobolev 1961, p. 16); and also Euclidean spaces R_n with a finite number of dimensions.

A set $M \subset E$ is *convex* if, for any two points u and v belonging to M, the straight-line segment connecting these points also belongs to M: i.e.

$$tu + (1-t)v \in M \quad \text{if} \quad 0 \leqslant t \leqslant 1.$$

A closed convex set $K \subset E$ is called a *cone* if the following conditions hold: (i) if $u \in K$, then $\alpha u \in K$ for all $\alpha \geqslant 0$; and (ii) for each pair of points u and $-u$ ($u \neq \theta$) belonging to E, at least one does not belong to K (K (*b*), p. 240; K (*c*), p. 17; Liusternik & Sobolev 1961, p. 128). Thus a cone comprises a bundle of rays originating from the zero point θ, no two of these rays having opposite directions. The collections of non-negative (or of non-positive) functions in the spaces C and L_p are the best-known examples of cones, but many other cones can be defined in these spaces. In § 6 we shall consider a cone in C which is narrower than the cone of non-negative functions. In the finite-dimensional space R_n, any one of the 2^n quadrants serves as an example of a cone. Already used in § 1 to illustrate principles, the set of non-negative numbers considered as a subset of R_1 is the most rudimentary example of a cone. Elements of a cone will be called *positive* elements or positive vectors.

A cone is said to be *solid* if it contains interior points, i.e. if it contains some sphere completely. The cone of non-negative functions in the space C is solid: any positive constant, for example, amounts to an interior point. In the spaces L_p, however, cones of non-negative functions are not solid. Various examples of solid and non-solid cones in finite-dimensional spaces can easily be defined: any cone in R_n will be a non-solid cone in R_m with $m > n$.

[This is a suitable place to cover a detail arising in applications of the subsequent theory. It is often the case that a solution is a continuous function which vanishes on the boundary ∂D of the domain of definition $\bar{D}$ (e.g. at the end-points of [0, 1]) but is positive on the open domain D. The solution may be established in the first place as an element of the cone of non-negative functions in C, but such a function is evidently not an interior element of the cone and this fact prevents the direct use of certain propositions. The following idea may then be useful. Consider the collection of continuous functions $v(y)$ vanishing on ∂D and such that

$$\|v\|_V = \max_{y\in\bar{D}} \frac{|v(y)|}{V(y)} < \infty, \tag{2.1}$$

where $V(y)$ is a given continuous function that satisfies $V > 0$ on D, $V = 0$ on ∂D, and whose inward normal derivative is positive everywhere on ∂D (e.g. take $V = y(1-y)$ if $\bar{D}$ is [0, 1]). This collection of functions becomes a Banach space under the norm (2.1). Functions $v(y)$ that are upwardly convex (see § 6.4) form a *solid* cone in this space, whereas the cone formed by such functions in C has no interior.]

A cone K is called *reproducing* if every element $u \in E$ is expressible in the form $u = v - w$ with $v, w \in K$. Any solid cone is reproducing, and cones of non-negative functions in the spaces L_p are also reproducing (K (*c*), p. 17). A cone K is called *normal* if a number $\delta > 0$ exists such that $\|u+v\| \geqslant \delta$ for any $u, v \in K$ with $\|u\| = \|v\| = 1$ (K (*c*), § 1.2; Liusternik & Sobolev 1961, p. 128). A necessary and sufficient condition for the normality of a cone is that the norm in E be semi-monotonic, which means that for arbitrary $u, v \in K$ the statement $v - u \in K$ implies $\|u\| \leqslant N\|v\|$, where N is a constant independent of u and v (K (*c*), p. 24). The norm is called monotonic if $N = 1$ in the preceding inequality. Cones of non-negative functions in the spaces C and L_p are normal, which follows from the obvious fact that the norms in these spaces are monotonic.

The specification of a cone K allows a *partial ordering* of the space E, which means that for certain pairs of elements $u, v \in E$ the ordering relation $u \leqslant v$ is defined and the symbol $\leqslant$ implies the usual properties associated with this symbol.† The axiomatic conditions of a partial ordering are provided by writing $u \leqslant v$ if $v - u \in K$ (K (*b*), p. 240; K (*c*), § 1.1.3). Then, in particular, $u \geqslant \theta$ if $u \in K$. When K is a cone of non-negative functions in C or L_p, the partial ordering has a simple interpretation. In the case of C, the notation $u \leqslant v$ means that $u(y) \leqslant v(y)$ (in the usual sense) for all values of the independent variable y; and in the case of L_p, the implication is that $u(y) \leqslant v(y)$ for almost all values of y. Unless a cone of non-negative functions is specifically in question, however, subsequent statements involving the symbol $\leqslant$ rest on the axiomatic definition.

An operator $\mathbf{A}$ acting in the space E (i.e. $\mathbf{A}u \in E$ if $u \in E$) is called *completely continuous* if it is continuous and transforms every bounded subset of E into a relatively compact set—that is, a set in whose closure every sequence of elements contains a strongly (in norm) convergent subsequence (Liusternik & Sobolev 1961, p. 129). A less restrictive property similarly defined is that of an operator which is completely continuous on a certain subset of E, e.g. a cone. In finite-dimensional spaces every bounded set is relatively compact (the extended Bolzano–Weierstrass theorem), and so the continuity of an operator suffices for its complete continuity in the sense defined.

† (i) If $u \leqslant v$, then $\alpha u \leqslant \alpha v$ for $\alpha \geqslant 0$ and $\alpha u \geqslant \alpha v$ for $\alpha < 0$. (ii) If $u \leqslant v$ and $v \leqslant u$, then $u = v$. (iii) If $u_1 \leqslant v_1$ and $u_2 \leqslant v_2$, then $u_1 + u_2 \leqslant v_1 + v_2$. (iv) If $u \leqslant v$ and $v \leqslant w$, then $u \leqslant w$.

An operator A, acting in a space which is partially ordered by means of a cone K, is called *positive* if it maps K into itself, i.e. if

$$Au \in K \quad \text{for} \quad u \in K.$$

The operator is called *monotonic* on K if $u \leqslant v$ $(u, v \in K)$ implies that $Au \leqslant Av$. Linear positive operators are obviously monotonic, but nonlinear positive operators are not necessarily so.

A nonlinear operator A acting in the space E is said to be differentiable at the point ϕ in the direction ξ if the abstract function $A(\phi + \epsilon\xi)$, having values in E, is differentiable with respect to ϵ at $\epsilon = 0$. The operator is said to be differentiable in the Fréchet sense if the increment $A(\phi + h) - A\phi$ can be expressed in the form

$$A(\phi + h) - A\phi = A'(\phi)\, h + \omega(\phi, h) \quad (h \in E), \tag{2.2}$$

where $A'(\phi)$ is a linear operator and

$$\lim_{\|h\| \to 0} \frac{\omega(\phi, h)}{\|h\|} = \theta \tag{2.3}$$

(K (a), p. 36; K (b), p. 135; K (c), § 3.1.1). The linear operator $A'(\phi)$ is called a *strong Fréchet derivative* if in (2.3) a strong limit is implied (i.e. if the quotient on the left-hand side converges in the norm of E). Then in particular, if $h = \epsilon\xi$ with $\|\xi\| = 1$, the remainder ω has the property

$$\omega(\phi, \epsilon\xi) = o(\epsilon).$$

If A is completely continuous, then the strong Fréchet derivative $A'(\phi)$ is also completely continuous (K (b), p. 135).

For the purposes of some subsequent arguments it will appear sufficient that the operator A should be differentiable only in the directions of a cone K. If (2.2) and (2.3) are qualified by the restriction $h \in K$, then $A'(\phi)$ is described as the strong Fréchet derivative with respect to the cone K. The existence of this Fréchet derivative implies that the derivative with respect to the cone (according to the first, simpler definition) also exists and the two derivatives are the same (K (c), § 3.1.2). This idea bears particularly on the question of linearizing the equation $\phi = A\phi$ in the neighbourhood of K close to the zero point θ. We are concerned with operators such that $A\theta = \theta$, and hence we may say that $\phi = A\phi$ has an *infinitesimal solution in* K if the linear equation

$$\xi = A'(\theta)\, \xi$$

has a non-zero solution in K.

Suppose that

$$Au = A'(\infty)\, u + \omega(u) \quad (u \in K), \tag{2.4}$$

where $A'(\infty)$ is a linear operator. This operator is called a *strong asymptotic derivative with respect to the cone* K if, for $u \in K$,

$$\lim_{\|u\| \to \infty} \frac{\|\omega(u)\|}{\|u\|} = 0 \tag{2.5}$$

(K (c), § 3.2.1).

If a number λ_θ exists such that the linear equation

$$\lambda_\theta \xi = A'(\theta)\, \xi \tag{2.6}$$

has a solution $\xi \in K$ $(\xi \neq \theta)$, we say that the operator $A'(\theta)$ has a positive *eigenvector* ξ corresponding to the *eigenvalue* λ_θ. With the same meaning we shall refer later to positive and other eigenvectors and corresponding eigenvalues of the asymptotic derivative $A'(\infty)$ and of the derivative operator $A'(\phi)$ at a particular non-zero point ϕ.

A wide class of linear operators, to which $A'(\phi)$ may belong, is defined as follows (K (*b*), ch. 5, § 5; K (*c*), ch. 2). Let u_0 be a given non-zero element of K. A linear positive operator $\boldsymbol{B}$ is called u_0-bounded (or, in K (*c*), u_0-positive) if for every $u \in K$ $(u \neq \theta)$, a positive integer n and positive numbers α, β can be found such that

$$\alpha u_0 \leqslant \boldsymbol{B}^n u \leqslant \beta u_0.$$

Operators in this class have the following properties when the cone K is reproducing. (i) If $\boldsymbol{B}$ is completely continuous, there exists an eigenvalue to which a positive eigenvector corresponds. (ii) Such an eigenvalue is always *simple* (i.e. no other eigenvector corresponds to it). (iii) There cannot be more than one such eigenvalue. (iv) This eigenvalue is greater in absolute magnitude than any other eigenvalue. It is helpful as regards the hydrodynamical problem that, in respect of operators that are positive on a cone of non-negative functions, the class includes linear integral operators whose (non-negative) kernels are the Green functions for boundary-value problems of the Sturm–Liouville type (see § 6 and appendix 2).

2.2. *Fixed-point theorems*

A solution $\phi \in K$ of the equation

$$\phi = A\phi, \tag{2.7}$$

where A is a positive nonlinear operator, amounts to a fixed point of the mapping by A of K into itself. Various theorems concerning the existence of non-zero fixed points in a cone have been proved by Krasnosel'skii, and we now state four of them which collectively appear to cover the general conjugate-flow problem (see § 3.2). For proofs of these theorems reference may be made to Krasnosel'skii's monograph (K (*c*), ch. 4), but we note that they are virtually proved, by considerably simpler arguments, in appendix 1 to this paper.

In each of the following theorems A is assumed to be a positive completely continuous operator on K such that $A\theta = \theta$.

Theorem I (K (*c*), theorem 4.12; see also § 4.4.4). *Let*

$$u - Au \notin K \quad \textit{if} \quad u \in K, \quad \|u\| = r > 0, \tag{2.8}$$

and

$$Au - u \notin K \quad \textit{if} \quad u \in K, \quad \|u\| = R > r. \tag{2.9}$$

Then A has at least one fixed point ϕ in K such that $r < \|\phi\| < R$.

Krasnosel'skii proved this theorem under a slightly weaker condition in place of (2.9), namely that $Au - (1+\epsilon)\, u \notin K$ for all $\epsilon > 0$ if $u \in K$, $\|u\| = R$ [a similar modification may be made to (2.10) below]. But this refinement does not seem worth inclusion for the purposes of the conjugate-flow theory in § 3. It is usually the case that the conditions (2.8) and (2.9) hold also for $\|u\| < r$ and $\|u\| > R$ respectively, although this is not essential to the theorem, and the description given by Krasnosel'skii to this case is that the operator A 'compresses' the cone.

Theorem II (K (*c*), theorem 4.14; see also § 4.4.4). *Let*

$$Au - u \notin K \quad \textit{if} \quad u \in K, \quad \|u\| = r > 0, \tag{2.10}$$

and

$$u - Au \notin K \quad \textit{if} \quad u \in K, \quad \|u\| = R > r. \tag{2.11}$$

Then A has at least one fixed point ϕ in K such that $r < \|\phi\| < R$.

In the usual case that the conditions (2.10) and (2.11) hold also for $\|u\| < r$ and $\|u\| > R$ respectively, Krasnosel'skii's description is that A 'expands' the cone.

THEOREM III (K (*c*), theorem 4.11). *Let* $\boldsymbol{A}$ *have a strong Fréchet derivative* $\boldsymbol{A}'(\theta)$ *and a strong asymptotic derivative* $\boldsymbol{A}'(\infty)$ *with respect to the cone K. Let* $\boldsymbol{A}'(\infty)$ *not have eigenvalues greater than or equal to unity in magnitude. Let* $\boldsymbol{A}'(\theta)$ *have a positive eigenvector corresponding to an eigenvalue* $\lambda_\theta > 1$ *[see equation* (2.6)*], and let* $\boldsymbol{A}'(\theta)$ *not have a positive eigenvector corresponding to an eigenvalue equal to unity. Then* $\boldsymbol{A}$ *has at least one non-zero fixed point in K.*

THEOREM IV (K (*c*), theorem 4.16). *Let the first condition of theorem* III *be satisfied. Let* $\boldsymbol{A}'(\theta)$ *not have a positive eigenvector corresponding to an eigenvalue greater than or equal to unity. Let* $\boldsymbol{A}'(\infty)$ *have a positive eigenvector corresponding to an eigenvalue* $\lambda_\infty > 1$, *and let* $\boldsymbol{A}'(\infty)$ *not have a positive eigenvector corresponding to an eigenvalue of unity. Then* $\boldsymbol{A}$ *has at least one non-zero fixed point in K.*

Theorems I and III are often interchangeable in applications, as also are II and IV. In fact, the conditions (2.8) and (2.9) of theorem I may in various ways be inferred from the conditions of III, if r is taken to be sufficiently small and R sufficiently large (see K (*c*), § 3.3); and similarly the conditions (2.10) and (2.11) of II may be inferred from the conditions of IV (K (*c*), § 3.4). A simple example of this will be given in § 5. It can be shown directly, moreover, that the existence of a non-zero fixed point in K is guaranteed if these various conditions are appropriately recombined: that is, if the restriction on the behaviour of the operator $\boldsymbol{A}$ near the point θ is taken from one of the theorems I and III (or II and IV) and is combined with the restriction from the other theorem on the behaviour of $\boldsymbol{A}$ for elements with large norms [K (*c*), p. 140; see also appendix 1 to the present paper].

We observe that all of these fixed-point theorems become obvious statements in respect of the one-dimensional space R_1. The rudimentary version of theorem III and theorem IV was noted in § 1.

2.3. *The rotation of a completely continuous vector field*

Let T denote some bounded domain of the Banach space E, and Γ the boundary of T ($\overline{T} = T \cup \Gamma$). We shall assume that the intersection of Γ with any finite-dimensional subspace admits triangulation (e.g. Γ is a sphere). In describing $\boldsymbol{\Phi}$ as a vector field given on $\overline{T}$, we mean that to every $u \in \overline{T}$ there corresponds a vector $\boldsymbol{\Phi}(u)$ from the space E. We particularly consider vector fields in the form $\boldsymbol{\Phi} = \boldsymbol{I} - \boldsymbol{A}$, where $\boldsymbol{I}$ is the identity operator and $\boldsymbol{A}$ is a completely continuous operator. Such fields are called completely continuous (K (*a*), p. 371; K (*b*), p. 105).

An important concept now to be recalled is the *rotation* (or topological degree) of a vector field $\boldsymbol{\Phi}$ on the boundary Γ of T. It is assumed that $\boldsymbol{\Phi}$ does not include the zero vector on Γ [i.e. $\boldsymbol{\Phi}(u) \neq \theta$ if $u \in \Gamma$]. Then, in the case of *finite-dimensional* spaces, the rotation may be defined as the degree (power) γ of the mapping

$$\boldsymbol{\Phi}_1(u) = \frac{\boldsymbol{\Phi}(u)}{\|\boldsymbol{\Phi}(u)\|} \quad (u \in \Gamma)$$

of $\boldsymbol{\Phi}$ into the unit sphere (i.e. the set $\|u\| = 1$, $u \in E$). For a basic explanation of this statement, reference can be made to books on topological methods [see, for example, Aleksandrov 1960; K (*b*), ch. 2, §§ 1 and 2; also Berger & Berger 1968, § 3.1 (iii)]. The definition may be extended to completely continuous vector fields in infinite-dimensional Banach spaces by considering finite-dimensional subspaces E_n, whose n elements can be used to approximate vectors $u \in \Gamma$ with a certain accuracy improving with n. If $\boldsymbol{P}_n$ is defined as the operator that projects the relatively compact set $\boldsymbol{A}(\Gamma)$ on E_n, the degree γ_n of the mapping

$$\frac{u - \boldsymbol{P}_n \boldsymbol{A} u}{\|u - \boldsymbol{P}_n \boldsymbol{A} u\|} \quad (u \in E_n \cap \Gamma)$$

is found to be independent of the selection of the approximating subspace provided n is large enough, i.e. $\gamma_n = \gamma_N$ if $n \geqslant N$. Accordingly, γ_N is established as defining the rotation $\gamma(\Gamma)$ of the completely continuous vector field $\boldsymbol{I}-\boldsymbol{A}$ on Γ (see K (*b*), pp. 105–108; Berger & Berger 1968, § 3.1).

The following simple facts are useful. (i) The rotation on Γ of the vector field $\boldsymbol{\Phi}(u) \equiv u-u^*$, where u^* is a fixed vector in E ($u^* \notin \Gamma$), is equal to unity if $u^* \in T$ and equal to zero if $u^* \notin T$. (ii) The rotation of a field $\boldsymbol{\Phi}$ is equal to zero if the directions of the vectors $\boldsymbol{\Phi}(u)$ for $u \in \Gamma$ are all different from the direction of a given vector $u^* \in E$, i.e. if none of the vectors $\boldsymbol{\Phi}(u)$ equals au^*, where a is a positive constant.

Two completely continuous vector fields $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ defined on Γ are said to be homotopic if for $u \in \Gamma$, $0 \leqslant t \leqslant 1$ an operator $\boldsymbol{X}(u,t)$ exists which is completely continuous as a mapping of the topological product $\Gamma \times [0,1]$ into E, and which is such that the completely continuous fields $\boldsymbol{F}(u,t) = u-\boldsymbol{X}(u,t)$ on Γ satisfy

$$\left.\begin{aligned} &\boldsymbol{F}(u,t) \neq \theta \quad (0 \leqslant t \leqslant 1),\\ \text{and}\quad &\boldsymbol{F}(u,0) = \boldsymbol{\Phi}(u), \quad \boldsymbol{F}(u,1) = \boldsymbol{\Psi}(u).\end{aligned}\right\} \tag{2.12}$$

It can be shown that *completely continuous vector fields that are homotopic have the same rotation* (K (*b*), p. 108; Berger & Berger 1968, theorems 3–8).

A point ϕ at which the vector field $\boldsymbol{\Phi} = \boldsymbol{I}-\boldsymbol{A}$ vanishes is a solution of the equation $\phi = \boldsymbol{A}\phi$, and as already explained we describe it as a fixed point. A well-known and extremely useful principle due to Leray & Schauder is that if the completely continuous field $\boldsymbol{\Phi}$ does not vanish on Γ, the boundary of T, and if the rotation of $\boldsymbol{\Phi}$ on Γ is not zero, then at least one fixed point of $\boldsymbol{\Phi}$ exists in T (K (*a*), p. 373; K (*b*), p. 123). It will be pointed out later, in § 3.4, that the fixed-point theorems stated in § 2.2 can be considered as instances of this principle.

Let ϕ_1 be an isolated fixed point of the field $\boldsymbol{\Phi}$, in the sense that no other fixed point lies in some neighbourhood of ϕ_1. Then the field $\boldsymbol{\Phi}$ will not vanish on any sphere of sufficiently small size centred on ϕ_1, and the rotation γ_1 of the field on every such sphere will be the same. This rotation is called the *index* of the fixed point. Now suppose that the field $\boldsymbol{\Phi}$ is completely continuous on $T \cup \Gamma$, has no fixed point on Γ, and has isolated fixed points $\phi_1, \phi_2 \ldots, \phi_\kappa$ in T, where the number k of fixed points is necessarily finite (K (*b*), p. 109). Then, according to a fundamental theorem established by Leray & Schauder, the sum of their indices equals the rotation of $\boldsymbol{\Phi}$ on Γ, thus

$$\gamma_1+\gamma_2+\ldots+\gamma_k = \gamma(\Gamma) \tag{2.13}$$

(K (*a*), p. 379; K (*b*), p. 109).

The following more general principle, which evidently includes (2.13), will also be used later. Let T_j ($j = 1,2,\ldots,m$) be mutually non-intersecting sub-domains of T, with boundaries Γ_j ($j = 1,2,\ldots,m$). Suppose that the vector field $\boldsymbol{\Phi}$ is completely continuous on $T \cup \Gamma$ and has fixed points only in the T_j, none being on their boundaries. Since $\boldsymbol{\Phi}$ does not vanish on Γ or on the Γ_j, the rotations $\gamma(\Gamma)$ and $\gamma(\Gamma_j)$ are defined, and we have

$$\gamma(\Gamma) = \gamma(\Gamma_1)+\gamma(\Gamma_2)+\ldots+\gamma(\Gamma_m). \tag{2.14}$$

2.4. *Evaluation of the index of a fixed point*

To find the index γ of a fixed point, use can be made of the fact that homotopic completely continuous vector fields have the same rotation. Let ϕ be a fixed point of the completely continuous field $\boldsymbol{\Phi} = \boldsymbol{I}-\boldsymbol{A}$, and let the operator $\boldsymbol{A}$ have a strong Fréchet derivative $\boldsymbol{A}'(\phi)$ at this point. If we assume that unity is not an eigenvalue of the linear operator $\boldsymbol{A}'(\phi)$, then it is easy to prove—as can be expected from the geometrical interpretation of the derivative—that ϕ is an isolated fixed point. Also, putting $u = \phi + h$, one can show that $\boldsymbol{\Phi}$ is homotopic to the linear completely continuous field $\boldsymbol{I} - \boldsymbol{A}'(\phi)$ on a sphere $\|h\| = \rho$ with ρ sufficiently small (K (*b*), pp. 136, 137). A simple case is presented when $\boldsymbol{A}'(\phi)$ does not have positive eigenvalues either equal to or greater than unity, for then the completely continuous fields

$$\boldsymbol{F}(h,t) = h - t\boldsymbol{A}'(\phi)\,h \quad (0 \leqslant t \leqslant 1)$$

evidently have no zero vector on the sphere. Thus $\boldsymbol{I} - \boldsymbol{A}'(\phi)$ is homotopic to $\boldsymbol{I}$, whose rotation is unity, and therefore $\gamma = 1$.

The case when $\boldsymbol{A}'(\phi)$ has eigenvalues greater than unity is more subtle but is covered, together with the preceding case, by the following theorem which again is due to Leray & Schauder (K (*a*), p. 375; K (*b*), p. 136; Rothe 1951, theorem 5.1). (The essentials of the usual proof of this theorem will be recalled for a special purpose at the end of § 3.4.) Assuming conditions as stated in the last paragraph, in particular that unity is not an eigenvalue of $\boldsymbol{A}'(\phi)$, we have that the index of the isolated fixed point ϕ is given by

$$\gamma = (-1)^{\beta}, \tag{2.15}$$

where β is the sum of the multiplicities of the eigenvalues of $\boldsymbol{A}'(\phi)$ that are greater than unity. (Here the term multiplicity means the number of linearly independent eigenvectors corresponding to a particular eigenvalue.) If the linear operator $\boldsymbol{A}'(\phi)$ has only simple eigenvalues (i.e. of unit multiplicity), then β in (2.15) is just the number of eigenvalues greater than unity. Such is the case for integral operators whose kernels are the Green functions for boundary-value problems of the Sturm–Liouville type (cf. last paragraph of § 2.1).

In the exceptional case when unity is an eigenvalue of $\boldsymbol{A}'(\phi)$ at a fixed point ϕ, the index of ϕ may still be calculable. Indeed, this case is of central interest in the theory of bifurcation points for nonlinear operators, and many results are available. In developing the conjugate-flow theory, however, we shall only need to consider the indices of non-zero fixed points in a cone (§ 3.4), and so the exceptional case appears to have no fundamental importance. It might reasonably be excluded by assumption, but the following proposition will be useful in extending the range of our conclusions (see K (*a*), pp. 376–378; K (*b*), ch. 4, § 4, particularly p. 223). Let $\boldsymbol{A}'(\phi)$ have an eigenvalue equal to unity, and suppose that in the neighbourhood of the fixed point ϕ the operator $\boldsymbol{A}$ is representable in the form

$$\boldsymbol{A}(\phi + h) = \boldsymbol{A}\phi + \boldsymbol{A}'(\phi)\,h + \boldsymbol{C}h + \boldsymbol{D}h \quad (h \in E),$$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are completely continuous operators satisfying the following conditions. First, $\boldsymbol{C}$ is homogeneous, thus

$$\boldsymbol{C}(\alpha h) = \alpha^{s}\boldsymbol{C}h \quad (\alpha = \text{const.}),$$

where s is an integer greater than unity. Secondly, $\boldsymbol{C}$ satisfies a Lipschitz condition

$$\|Ch_1 - Ch_2\| \leqslant q(\rho)\,\|h_1 - h_2\| \quad \text{if} \quad \|h_1\|,\ \|h_2\| \leqslant \rho,$$

where $q(\rho)$ is such that the quotient $q(\rho)/\rho^{s-1}$ remains bounded in the limit as $\rho \to 0$. Finally, the operator $\boldsymbol{D}$ is of an order of smallness higher than s, i.e.

$$\|\boldsymbol{D}h\| = o(\|h\|^{s}).$$

(Completely continuous integral operators with sufficiently smooth integrands usually satisfy these conditions: for examples see the references given above.) Let the eigenvalue unity be simple. Then, if s is even, the index γ of the fixed point ϕ is always *zero*. If s is odd it turns out that $|\gamma| = 1$, and an easily applicable criterion deciding the sign of γ can be given; but we shall not need this result.

When ϕ is a non-zero fixed point, it is evidently to be expected that $s = 2$ (and so $\gamma = 0$) almost always. That is, the approximation of the operator $\boldsymbol{A}$ in the neighbourhood of ϕ will generally present $\boldsymbol{C}$ as a quadratic operator. We thus appreciate that the possibility of unity being an eigenvalue of $\boldsymbol{A}'(\phi)$ and *in addition* $s > 2$ is a great deal more extraordinary than the possibility of the first property arising alone.† With considerably extended generality, therefore, the exceptional case in question may be allowed on the assumption that, if it does arise, the index of the respective fixed point is zero.

2.5. *Variational methods*

Let E be specifically the real Hilbert space $L_2(D)$. The inner product of any two elements $u(y), v(y) \in L_2(D)$ is defined as

$$\langle u, v\rangle = \int_D uv\,\mathrm{d}y, \tag{2.16}$$

and the norm in this space is given by

$$\|u\|_{L_2} = \langle u, u\rangle^{\frac{1}{2}}. \tag{2.17}$$

The functional (scalar) $\mathscr{F}(u)$, defined on some open set $T \subset E$, is said to have a linear Gateaux differential if for all $u \in T$ and all $h \in E$ the fomula

$$\lim_{t\to 0}\left\{\frac{\mathscr{F}(u+th)-\mathscr{F}(u)}{t}\right\} = \langle \boldsymbol{\Phi} u, h\rangle \tag{2.18}$$

defines the expression on the right-hand side as a linear functional of h. The operator $\boldsymbol{\Phi}$ is called the *gradient* of the functional $\mathscr{F}$, and we write $\boldsymbol{\Phi} = \operatorname{grad}\mathscr{F}$ (Vainberg 1964, p. 54). The gradient is said to be strong if $\mathscr{F}$ has a Fréchet differential: that is, if the scalar remainder

$$\mathscr{F}(u+th)-\mathscr{F}(u)-t\langle \boldsymbol{\Phi} u, h\rangle \quad \text{is} \quad o(t\|h\|).$$

Operators definable in this way are called *potential* operators.

[We note that the gradient of a scalar function of position in the finite-dimensional space R_n is definable in just the same way. The inner product in R_n is

$$\langle u, v\rangle = \sum_{i=1}^{n} u_i v_i, \tag{2.19}$$

where u_i and v_i $(i = 1, 2, \ldots, n)$ are the orthogonal components of the vectors u and v; and again the norm is the positive square-root of $\langle u, v\rangle$. Everything we shall say about variational principles with respect to L_2 has a counterpart with respect to R_n.

We also note, incidentally, that a linear operator $\boldsymbol{B}$ acting in L_2 is called *self-adjoint* if

$$\langle \boldsymbol{B}u, v\rangle = \langle u, \boldsymbol{B}v\rangle. \tag{2.20}$$

When a linear operator has the form $\bar{\boldsymbol{B}}u = \boldsymbol{B}(fu)$, where $\boldsymbol{B}$ satisfies (2.20) and f is a bounded positive function, we shall describe $\bar{\boldsymbol{B}}$ as *self-adjoint with respect to the weight function* f; for

† This idea is clearly illustrated by the elementary example considered in § 1, where ϕ is a scalar variable satisfying $\phi = A(\phi)$. The first exceptional property is that $A'(\phi) = 1$, and the additional property is that $A''(\phi) = 0$. Figure 1 (*c*) exemplifies the less extraordinary case of a fixed point with zero index [i.e. $A'(\phi) = 1$ but $A''(\phi) \neq 0$].

evidently the preceding definition of self-adjointness applies again with the modified inner product

$$\langle u, v\rangle_f = \int_D fuv \, dy. \tag{2.21}$$

It is easily seen that the self-adjoint linear operator $\boldsymbol{B}$ is the gradient of the functional $\frac{1}{2}\langle u, \boldsymbol{B}u\rangle$. The self-adjointness of a linear matrix transformation in R_n may be similarly defined, by juxtaposing (2.20) with the definition (2.19) of the inner product in R_n.]

In § 3.5 the hydrodynamical problem will be considered recast so that a function ζ representing a conjugate flow is a solution of

$$\zeta = \boldsymbol{G}\zeta, \tag{2.22}$$

where $\boldsymbol{G}$ is a completely continuous potential operator, being the gradient of a functional Ω. The identity operator $\boldsymbol{I}$ is the (strong) gradient of $\frac{1}{2}\langle u, u\rangle$ (Vainberg 1964, p. 55). Hence the completely continuous vector field $\boldsymbol{I}-\boldsymbol{G}$ may be considered as the gradient field of the functional

$$\Lambda(u) = \tfrac{1}{2}\langle u, u\rangle - \Omega(u), \tag{2.23}$$

and a solution $\zeta \in L_2$ of (2.22) amounts to a stationary point of Λ, i.e. grad $\Lambda(\zeta) = \theta$.

Let $\sigma \subset L_2$ be a bounded set which is *weakly closed*: this means that σ contains all its weak limit points.† With regard to the abstract nonlinear functional Λ, suppose it can be shown (*a*) that Λ is strongly differentiable on σ (i.e. $\boldsymbol{G}$ is defined on σ), (*b*) that Λ has a lower bound on σ which is achieved at a point $\zeta \in \sigma$, and (*c*) that ζ is not on the boundary of σ. The fact (*b*) means that $\Lambda(\zeta)$ is a minimum. But, in consequence of (*a*) and (*c*), a necessary condition for a minimum at ζ is that this point be stationary (see Vainberg 1964, p. 77, theorem 9.1: it is helpful to recall the familiar counterpart of this proposition in the analysis of differentiable scalar functions of a finite-dimensional vector). Thus (*a*), (*b*) and (*c*) together imply that (2.22) has a solution to which a minimum value of Λ corresponds. If, however, (2.22) has the trivial solution $\zeta = \theta$ and $\theta \in \sigma$, then obviously a useful existence proof must establish another fact, namely (*d*) that Λ has values less than $\Lambda(\theta)$ on σ.

In this approach the assumption that the nonlinear operator is completely continuous is again crucial, being required to establish (*b*). In a Hilbert space any weakly closed, bounded set is weakly compact (Vainberg 1964, p. 11 and p. 23, theorem 1.7). Hence one can show that if the functional Λ is *weakly lower semi-continuous*‡ on σ, then it has a lower bound which it achieves on σ (Vainberg, p. 68, theorem 9.2). The functional $\langle u, u\rangle$ has this property, and so Λ has it also if the functional Ω is *weakly continuous*§ on σ, which property is provided if $\boldsymbol{G}$ is a completely continuous operator (Vainberg, p. 76, theorem 8.2).

In the main development of the conjugate-flow theory, variational methods will not be used primarily to establish the existence of solutions. The need to order the physical possibilities' which is met by specifying solutions in a cone, seems to mark out the use of topological methods as the most natural general approach to the existence problem.‖ In § 6.5, however, the varia-

† If a subset of a Banach space is convex, then its weak closure coincides with its strong closure (Kelley, Namioka *et al.* 1963, p. 154, theorem 17.1). For example, in any Banach space the sphere $\|u\| \leqslant r$ is weakly closed.

‡ This means that if $\{u_m\}$ $(m = 1, 2, \ldots)$ is any sequence of points in σ such that $u_m \to v$ *weakly* as $m \to \infty$, then $\liminf_{m\to\infty} \Lambda(u_m) \geqslant \Lambda(v)$.

§ This means, with reference to the preceding footnote, that $\lim_{m\to\infty} \Omega(u_m) = \Omega(v)$.

‖ There are, of course, many connexions between the two approaches, some of which will be touched upon in § 3.5. Reference may be made to papers by Rothe (1951, 1952) for studies of the Leray–Schauder indices of fixed points and other topological properties of completely continuous vector fields that are the gradients of functionals in Hilbert space.

tional argument just outlined will be applied to an example, for which a special device will be used to restrict solutions to a cone of non-negative functions.

The main application of variational ideas in the present context is to the theory of flow-force differences between conjugate flows (§ 3.5). In specific examples the matter can sometimes be treated by other methods (see, for example, Fraenkel 1967), but the use of variational ideas seems essential to the interpretation of certain general principles concerning flow force.

3. Abstract theory of conjugate flows

Recapitulating (1.2), we take the equation that governs steady x-independent flows to be

$$\phi = \boldsymbol{A}(\mu)\,\phi. \tag{3.1}$$

Here $\boldsymbol{A}(\mu)$ stands for a nonlinear operator such that $\boldsymbol{A}(\mu)\,\theta = \theta$ and depending on the positive parameter μ, which varies inversely with the scale of the flow velocity in the x-direction. We have in view typically that if this velocity is a constant c, or if a scale factor c is attached to a non-uniform velocity distribution, then $\mu = a/c^2$, where a is a positive constant. Thus a scaling down of the flow velocity, such as to produce a critical state from a supercritical one, corresponds to an increase in μ. We assume that, in a sense to be made precise presently, the operator $\boldsymbol{A}(\mu)$ is increasing with μ: as a prototype we can consider $\boldsymbol{A}(\mu) \equiv \mu\hat{\boldsymbol{A}}$, where $\hat{\boldsymbol{A}}$ is an operator independent of the velocity scale. The opposite case, whose prototype is $\boldsymbol{A}(\mu) \equiv \mu^{-1}\hat{\boldsymbol{A}}$, may be presented by the most convenient formulation of the hydrodynamical problem† (e.g. examples 1 and 2, §§ 4 and 5); but the modifications then needed in the following statements are obvious, and so the present assumption is warranted for the sake of concision. The dependence of operators on the parameter μ will often be left implicit when not bearing directly on the discussion: for instance, the right-hand side of (3.1) may be written as $\boldsymbol{A}\phi$.

It is worth further emphasis that the following arguments cover many different specific forms of the hydrodynamical problem. The solution ϕ of (3.1) may be a vector in a finite-dimensional or infinite-dimensional space, and various propositions are to be established without the need to distinguish between the spaces in which the problem may be posed. We assume that the operator $\boldsymbol{A}$ acts in some Banach space E and is completely continuous (or just continuous in the case of finite-dimensional spaces). This assumption is essential to the global existence theorems that will be brought to bear on the problem, but the arguments presented next in § 3.1, concerning the definition of supercritical and subcritical states of flow, do not depend on it.

3.1. *The supercritical–subcritical classification*

Generally (3.1) has many solutions corresponding to different, possibly overlapping ranges of the parameter μ. An ordering of the possibilities thus represented is needed to make sense of the physical problem, and a natural approach would be to characterize them in the same way as the solutions of the linearized form of (3.1), by taking account, for example, of the number of zeros of ϕ on the open domain over which it is defined (cf. Benjamin 1966). In keeping with this idea we consider here the possibility of solutions in a cone K. In applications this is usually (but is not necessarily) a cone of non-negative functions, in which case typically the 'first wave mode' is

† Particularly when ϕ is a finite-dimensional vector, it may be just a question of convenience whether (3.1) or its inverse $\phi = \boldsymbol{A}^{-1}\phi$ is considered. If $\boldsymbol{A}(\mu)$ is decreasing with μ, then generally the inverse operator $\boldsymbol{A}^{-1}(\mu)$ will be increasing.

specified and the respective values of μ are smaller—and so velocities are larger—than for other modes (cf. K (*c*), § 2.4).†

We assume that A is a positive operator which has a strong Fréchet derivative with respect to the cone K, and that

$$A'(\phi;\mu_1)\,u \geqslant A'(\phi;\mu_2)\,u \quad \text{if} \quad \phi, u \in K, \quad \mu_1 \geqslant \mu_2 > 0. \tag{3.2}$$

We also assume for the time being that, for $\phi \in K$, $A'(\phi;\mu)$ has two of the properties noted in the last paragraph of § 2.1, being common to a wide class of linear operators: namely, (ii) an eigenvalue $\lambda_\phi > 0$ to which a positive eigenvector corresponds is simple, and (iii) such an eigenvalue is unique.

We next suppose that for a certain value of the velocity scale, such that $\mu = \mu_c$ say, equation (3.1) has an infinitesimal solution in K. This means that the derivative operator at the zero point of E, $A'(\theta;\mu_c)$, has a positive eigenvector corresponding to the eigenvalue unity, thus

$$\xi_c = A'(\theta;\mu_c)\,\xi_c, \quad \text{where} \quad \xi_c \in K. \tag{3.3}$$

The primary flow corresponding to $\mu = \mu_c$ is classified as *critical*, in accordance with the definition that an infinitesimal wave of extreme length can be superposed on it (see § 1).

We further suppose that for $\mu \neq \mu_c$, the operator $A'(\theta;\mu)$ has a positive eigenvector corresponding to an eigenvalue $\lambda_\theta > 0$, thus

$$\lambda_\theta \xi = A'(\theta;\mu)\,\xi, \quad \text{where} \quad \xi \in K. \tag{3.4}$$

Hence in various ways it may appear that

$$\lambda_\theta < 1 \quad \text{implies} \quad \mu < \mu_c, \quad \text{so the primary flow is } \textit{supercritical}, \tag{3.5a}$$

$$\lambda_\theta > 1 \quad \text{implies} \quad \mu > \mu_c, \quad \text{so the primary flow is } \textit{subcritical}, \tag{3.5b}$$

For instance, let us assume that a maximal positive number β_m can be defined such that‡

$$\xi \geqslant \beta_m \xi_c, \quad \text{but} \quad \xi - \beta\xi_c \notin K \quad \text{if} \quad \beta > \beta_m.$$

If the implication stated in (3.5*a*) is assumed not to be true, so that there exists $\mu \geqslant \mu_c$ when $\lambda_\theta < 1$, it follows by virtue of the condition (3.2) that

$$A'(\theta;\mu)\,\xi \geqslant A'(\theta;\mu_c)\,\xi. \tag{3.6}$$

† For the study of 'higher modes', another class of cones defined as follows would be useful. Suppose that the linearized form $A'(\theta;\mu)$ of the operator $A(\mu)$ has a set of normalized eigenvectors ξ_n ($\|\xi_n\| = 1$) corresponding to simple eigenvalues $\lambda_\theta^{(n)}$ ($n = 1, 2, \ldots$). Values of μ, say μ_n, for which $\lambda_\theta^{(n)} = 1$ may be bifurcation points for the nonlinear equation (3.1), so that when μ is close to a particular μ_n it may be expected that the equation has a solution not much different from $b_n \xi_n$, where b_n is a small constant. To make an exact study of this situation, it is suitable to consider the linear operator P_n that projects any element $u \in E$ onto the one-dimensional subspace defined by ξ_n (i.e. the straight line passing through the zero point θ and the point ξ_n). Thus we consider $P_n u = b_n(u)\,\xi_n$, where $b_n(u)$ is a linear functional, and we represent u as the sum $u = P_n u + P^n u$, where $P^n u$ is an element of the complementary subspace of E having unit defect. A cone in E is comprised by the collection of elements for which

$$b_n(u) \geqslant 0, \quad \|P^n u\| \leqslant \kappa \|P_n u\| = \kappa b_n(u),$$

where κ is a positive number (K(*c*), p. 35; Vainberg 1964, p. 91). This cone is easily seen to be normal.

It may turn out that the set of eigenvectors $\{\xi_n\}$ is a complete orthogonal basis in L_2 (see for example, § 6.3). Then $b_n(u)$ is just the coefficient of ξ_n in the orthogonal-series expansion of u: that is, $b_n(u) = \langle \xi_n, u\rangle$. It is readily seen, moreover, that the second of the above inequalities defining a cone is equivalent to

$$(1+\kappa^2)^{\frac{1}{2}} b_n \geqslant \left(\sum_{m=1}^{\infty} b_m^2\right)^{\frac{1}{2}} = \|u\|_{L_2}.$$

‡ The validity of this assumption is obvious if K is a solid cone and ξ is an interior element, but it can also be justified under considerably less restrictive conditions. If $A'(\theta;\mu)$ and $A'(\theta;\mu_c)$ are both u_0-bounded (see concluding paragraph of § 2.1), it is a simple matter to show that there exists a $\beta_m > 0$ defined as above.

The positiveness of the operator $\boldsymbol{A}$ on the cone K implies that $\boldsymbol{A}'(\theta;\mu_c)$ is also a positive operator (K (*c*), p. 103), and hence

$$\boldsymbol{A}'(\theta;\mu_c)\,\xi \geqslant \beta_m \boldsymbol{A}'(\theta;\mu_c)\,\xi_c = \beta_m\,\xi_c. \tag{3.7}$$

But the combination of (3.4), (3.6) and (3.7) shows that

$$\xi \geqslant (\beta_m/\lambda_\theta)\,\xi_c,$$

which contradicts the maximal property of β_m if $\lambda_\theta < 1$. Thus the truth of (3.5*a*) is established. An obvious adaptation of this argument may be used to prove (3.5*b*), and similarly the converse of (3.5*a*) and (3.5*b*) may be demonstrated (e.g. a supercritical primary flow implies $\lambda_\theta < 1$). Another form of argument, applying when the linear operator $\boldsymbol{A}'(\theta;\mu)$ is self-adjoint with respect to a positive weight function [see (2.21) and context], will be noted in § 6.4.

A conjugate flow represented by a non-trivial solution ϕ of (3.1) can be classified in the same way. Suppose that the derivative operator $\boldsymbol{A}'(\phi;\mu)$ at the point $\phi\in K$ has a unique positive eigenvector corresponding to an eigenvalue $\lambda_\phi > 0$, thus

$$\lambda_\phi\,\eta = \boldsymbol{A}'(\phi;\mu)\,\eta, \quad \text{where} \quad \eta\in K. \tag{3.8}$$

If $\boldsymbol{A}'(\phi;\mu)$ is a positive operator (which property is ensured if the operator $\boldsymbol{A}$ is monotonic on K), then the preceding form of argument can be used to show that the conjugate flow is supercritical or subcritical accordingly as $\lambda_\phi < 1$ or $\lambda_\phi > 1$. Otherwise, as will be shown in § 6.4, this classification may be established by using the self-adjointness of $\boldsymbol{A}'(\phi;\mu)$ with respect to a positive weight function, if it has this property.

As regards the classification of flows and also questions of existence, it is generally sufficient to assume that at points ϕ in the cone the nonlinear operator $\boldsymbol{A}$ is strongly differentiable in the directions of the cone. Later, however, in order to clarify certain deductions bearing on the transcritical property of conjugate flows, a stronger assumption is needed about the differentiability of $\boldsymbol{A}$. We shall suppose that at a solution point $\phi \in K(\phi \neq \theta)$, $\boldsymbol{A}'(\phi)$ has a strong Fréchet derivative in every direction of the space E, and we shall take account of eigenvectors of $\boldsymbol{A}'(\phi)$ that are not elements of K. It will then be convenient to depend on the fourth property of certain linear operators noted in the concluding paragraph of § 2.1, namely that the (simple) eigenvalue λ_ϕ to which a positive eigenvector corresponds is greater than any other eigenvalue.

The simplest view of the theory as a whole is given by taking $\boldsymbol{A}'(\phi)$ to be a u_0-bounded operator and K to be a reproducing cone, in which case the properties (i) to (iv) noted at the end of § 2.1 are automatically provided (see K (*c*), ch. 2): this still allows a very wide range of applicability. Extensions of the present arguments to physically meaningful examples in which the properties (i) to (iv) do not all hold appear feasible (e.g. concerning 'higher modes'), but discussion of them here would further complicate the presentation to an unwarranted extent. In passing, however, we may suitably note two propositions that establish convenient properties under alternative assumptions. If a linear operator $\boldsymbol{B}$ is u_0-bounded and K is a reproducing cone, it follows simply that a positive integer n and a positive number γ can be found such that $\boldsymbol{B}^n u \leqslant \gamma u_0$ for any particular $u \in E$. If $\boldsymbol{B}$ is completely continuous, the latter condition implies that there exists a simple eigenvalue to which a positive eigenvector corresponds, and which is greater in magnitude than any other eigenvalue of $\boldsymbol{B}$ (K (*b*), p. 263, theorem 2.3). But the original requirement that K is reproducing is not necessary to the latter condition, which might therefore be taken, together with the u_0-boundedness of $\boldsymbol{B}$, as a somewhat more widely applicable specification. If $\boldsymbol{B}$ is not u_0-bounded from below, the following proposition may be useful. Let

K be a reproducing cone, and let h_0 be a positive eigenvector of the linear operator $\boldsymbol{B}$ which is h_0-bounded from above (i.e. a $\beta > 0$ and an n can be found such that $\boldsymbol{B}^n u \leqslant \beta h_0$ for $u \in K$). Then none of the remaining eigenvalues of $\boldsymbol{B}$ is greater in magnitude than the eigenvalue to which h_0 corresponds (K (*c*), theorem 2.14).

3.2. *Some necessary conditions*

Incidentally to the main development of the theory, we note here some simple considerations showing that a non-zero positive solution of (3.1) is impossible in certain cases. The nonlinear positive operator $\boldsymbol{A}$ (whose dependence on the parameter μ is left implicit henceforth) will be called *concave* if

$$A\{tu_1+(1-t)\,u_2\} \geqslant tAu_1+(1-t)\,Au_2 \quad \text{for} \quad u_1, u_2 \in K\ (u_1 \neq u_2),\ 0 \leqslant t \leqslant 1, \tag{3.9}$$

and *convex* if

$$A\{tu_1+(1-t)\,u_2\} \leqslant tAu_1+(1-t)\,Au_2 \quad \text{for} \quad u_1, u_2 \in K\ (u_1 \neq u_2),\ 0 \leqslant t \leqslant 1. \tag{3.10}$$

These terms will be taken to imply, moreover, that neither $\geqslant$ in (3.9) not $\leqslant$ in (3.10) reduces to equality except at $t = 0$ and $t = 1$.

Let the cone be specifically the cone K_+ of non-negative functions in L_2. Also let the linear operator $\boldsymbol{A}'(\theta)$ be self-adjoint with respect to a positive weight function f [see (2.21) and context], which is a property often arising in applications (cf. § 6). Our aim is to show that in the case of a concave (respectively convex) operator $\boldsymbol{A}$, it is then necessary for the primary flow to be subcritical (respectively supercritical) if a conjugate flow is to exist represented by a non-zero solution of (3.1) in K_+. By the use of arguments akin to those in § 3.1, corresponding propositions can in fact be established under rather weak assumptions when the cone is arbitrary and $\boldsymbol{A}'(\theta)$ is not necessarily self-adjoint; but we shall pass over these generalizations.

We first assume that $\boldsymbol{A}$ is concave, and that $\phi \in K_+$ $(\phi \neq \theta)$ exists such that $\phi = \boldsymbol{A}\phi$. Putting $u_1 = \phi$ and $u_2 = \theta$ in (3.9), we obtain

$$\boldsymbol{A}(t\phi) \geqslant t\phi \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

It is further implied by the definition of concavity [see remark following (3.10)] that $\boldsymbol{A}(t\phi) - t\phi$ is a non-zero element of K_+ for $0 < t < 1$. Thus, taking any particular value t_0 such that $0 < t_0 < 1$, we have

$$\boldsymbol{A}(t_0\phi) \geqslant t_0\phi + t_0\chi \quad \text{with} \quad \chi \in K_+\ (\chi \neq \theta). \tag{3.11}$$

With $u_1 = t_0\phi$ and $u_2 = \theta$, (3.9) now gives

$$\frac{1}{t'}\boldsymbol{A}(t't_0\phi) \geqslant \boldsymbol{A}(t_0\phi) \quad (0 \leqslant t' \leqslant 1).$$

Taking the limit as $t' \to 0$ and combining the result with (3.11), we obtain

$$\boldsymbol{A}'(\theta)\,\phi \geqslant \phi + \chi. \tag{3.12}$$

Since the weight function f is positive, the inner product (2.21) of any two non-zero elements of K_+ must be positive unless their supports on D are disjoint. Considering the eigenvector $\xi \in K_+$ of $\boldsymbol{A}'(\theta)$ that corresponds to the eigenvalue λ_θ [see (3.4)], we assume that this function is positive almost everywhere. Hence, using the self-adjointness of $\boldsymbol{A}'(\theta)$, we have

$$\begin{aligned} 0 < \langle \chi, \xi \rangle_f \leqslant \langle \{\boldsymbol{A}'(\theta)\,\phi - \phi\}, \xi \rangle_f &= \langle \phi, \boldsymbol{A}'(\theta)\,\xi \rangle_f - \langle \phi, \xi \rangle_f \\ &= (\lambda_\theta - 1)\,\langle \phi, \xi \rangle_f, \end{aligned} \tag{3.13}$$

which shows that $\lambda_\theta > 1$. Thus the primary flow must be subcritical.

An obvious adaptation of this argument leads to the conclusion that the primary flow must be supercritical ($\lambda_\theta < 1$) if $\boldsymbol{A}$ is a convex operator and (3.1) has a non-zero solution in K_+.† It can similarly be seen that if $\boldsymbol{A}$ is concave (respectively convex), then a solution in the cone $-K_+$ of non-positive functions is impossible unless the primary flow is supercritical (respectively subcritical).

The same set of conclusions evidently holds when the solution ϕ of the hydrodynamical problem is a non-zero element of the finite-dimensional space R_n and the matrix transformation $\boldsymbol{A}'(\theta)$ is self-adjoint with respect to a positive weight. These conclusions also hold with regard to problems posed in the space C, for then the definition (2.21) is still meaningful as a linear functional of u if v is also an element of C.

3.3. *Existence of conjugate flows: example of the transcritical property*

In specific examples the existence of a non-trivial solution $\phi \in K$ of (3.1) may be established by one or other of the four fixed-point theorems stated in § 2.2. Either of the theorems I or III may be applicable when the primary flow is subcritical, and II or IV when it is supercritical. Consideration of the one-dimensional case (i.e. where ϕ is just a scalar, as was discussed in § 1) shows that these four theorems and combinations of them comprise the most general set of propositions assuring the existence of a non-zero fixed point of a positive completely continuous operator $\boldsymbol{A}$ such that $\boldsymbol{A}\theta = \theta$. Accordingly, a study of the transcritical property of conjugate flows can suitably be made by taking the conditions of these theorems as prescribed data. A comprehensive view of the possibilities will be developed in § 3.4, where the concept of the Leray–Schauder index will be used; but there are various special aspects of the problems that can be illuminated by arguments associated with the present fixed-point theorems. The following argument is helpful in that an outstanding aspect of the simple case considered in § 1 is generalized to spaces with an arbitrary number of dimensions.

We assume that the primary flow is subcritical so that $\lambda_\theta > 1$, and that $\boldsymbol{A}'(\theta)$ does not have a positive eigenvector corresponding to an eigenvalue of unity. (The latter condition is, of course, ensured if we suppose $\boldsymbol{A}'(\theta)$ to have the property (iii) noted at the end of § 2.1). Also $\boldsymbol{A}$ is a monotonic operator, and has a strong asymptotic derivative $\boldsymbol{A}'(\infty)$ with respect to the cone K. If no eigenvalue of the linear operator $\boldsymbol{A}'(\infty)$ is equal to or greater than unity, then all the conditions of theorem III are provided and it can be concluded that $\boldsymbol{A}$ has at least one non-zero fixed point ϕ in K.

The solution ϕ is not necessarily unique, and on the assumption that it is not we discriminate the solution—or solutions— satisfying

$$\phi - \phi^* \notin K, \tag{3.14}$$

where ϕ^* stands collectively for every other non-trivial solution of (3.1) in K. A solution that is distinctly closest to the point θ, in the sense that $\|\phi\| < \|\phi^*\|$, obviously satisfies this condition if the norm is monotonic on K (see remarks following the definition of a normal cone in §2.1); and there is always one among several solutions that satisfies it because, by the definition of a cone, either $\phi_1 - \phi_2$ or $\phi_2 - \phi_1$ must fall outside K if two points ϕ_1 and ϕ_2 are different. We shall now show that the respective conjugate flow cannot be subcritical: that is, if the possibility of the flow being exactly critical is excluded, then it must be supercritical. In view of what was explained

† A conclusion equivalent to this was established by Sheer (1968), who considered a specific problem of the general kind now in question but used a totally different approach.

in the context of (3.8), this amounts to showing that $\lambda_\phi < 1$ if $\lambda_\phi \neq 1$. (The general need to qualify such propositions by excluding the extraordinary possibility $\lambda_\phi = 1$ was pointed out in § 1.) In the following proof the assumption that $\lambda_\phi > 1$ is shown to lead to a contradiction.

We take the positive solutions of the linear equations (3.4) and (3.8) to be normalized, $\|\xi\| = \|\eta\| = 1$, and we assume K to be a solid cone in which ϕ, ξ and η are interior elements. [The latter assumption is made for simplicity, since it gives immediate justification for the inequalities (3.15) to (3.17) which follow, but the present argument can be adapted to the case of cones without interiors: cf. the second footnote to § 3.1.] We consider the *conical segment* $\mathscr{S}$ consisting of the collection of elements u that satisfy

$$\epsilon\xi \leqslant u \leqslant \phi - \delta\eta, \tag{3.15}$$

where ϵ and δ are small positive numbers. The set $\mathscr{S}$ is evidently convex, and it may be assumed not to be empty if ϵ and δ are sufficiently small. On the additional assumption that the cone K is normal, $\mathscr{S}$ can be considered as a closed set (cf. K (*c*), §§ 1.2, 1.3). Since $\lambda_\theta > 1$, we have

$$\begin{aligned} A(\epsilon\xi) &= \epsilon A'(\theta)\,\xi + o(\epsilon) \\ &= \lambda_\theta\,\epsilon\xi + o(\epsilon) \geqslant \epsilon\xi \end{aligned} \tag{3.16}$$

if ϵ is small enough. Also, on the assumption that $\lambda_\phi > 1$,

$$\begin{aligned} A(\phi - \delta\eta) &= A\phi - \delta A'(\phi)\,\eta + o(\delta) \\ &= \phi - \lambda_\phi\delta\eta + o(\delta) \leqslant \phi - \delta\eta \end{aligned} \tag{3.17}$$

if δ is small enough. The inequalities (3.16) and (3.17) imply that the monotonic operator A transforms $\mathscr{S}$ into itself. And, since A is taken to be completely continuous, it follows by virtue of Schauder's principle that A has a fixed point in $\mathscr{S}$ (K (*c*), § 4.1.1). But this is contradictory either if ϕ is unique or if ϕ is selected by the condition (3.14), and thus the assumption that $\lambda_\phi > 1$ is seen to be incorrect. We have shown that corresponding to the given subcritical primary flow, at least one conjugate flow is supercritical.

3.4. *Index theory*

The transcritical property of conjugate flows can be examined in a very general way by considering the Leray–Schauder indices of fixed points $\phi \in K$, and using the topological theorems explained in § 2.3 concerning the rotation of a completely continuous vector field. With this aim it is simplest (later on) to assume that K is a solid cone and the non-zero solutions ϕ are interior elements, but we shall finally show that out main conclusions can be generalized to include cases where the cone K has no interior.

We may define a completely continuous operator $\tilde{A}$ which maps the whole of the space E into K and is identical with the completely continuous positive operator A on K, that is

$$\tilde{A}u \in K \quad \text{if} \quad u \in E, \tag{3.18}$$

and

$$\tilde{A}u = Au \quad \text{if} \quad u \in K. \tag{3.19}$$

Such an extension of the operator A always exists, by virtue of the cone being by definition a closed and convex subset of E: proof of this fact is available from the theory of retracts (e.g. see Hu 1965, p. 57, theorem 14.1). For example, let K be a cone of non-negative functions in one of the spaces C or L_p, or let it be a narrower cone into which this cone is mapped by the operator A (see § 6.4). Then an operator with the required properties is $\tilde{A} = AC$, where C is the continuous operator taking any function $u \in E$ into the function

$$Cu = \tfrac{1}{2}(u + |u|),$$

which is the same as u where u has non-negative values and is zero elsewhere (cf. K (*b*), p. 248). Obviously, any fixed point of $\tilde{\boldsymbol{A}}$ must belong to K and is therefore also a fixed point of $\boldsymbol{A}$.

The subsequent interpretation of the transcritical property rests on the following propositions which complement the fixed-point theorems of § 2.2. Here S_r and S_R stand for spheres $\|u\| = r > 0$ and $\|u\| = R > r$ in E, and it is implied that $r < \|\phi\| < R$, where $\phi \in K$ is the solution of (3.1) whose existence is guaranteed by one or other of the fixed-point theorems.

Theorem A. *Let the conditions of theorem I be satisfied, or let the conditions of theorem III be satisfied and in addition let the cone K be normal.*† *Then there exist spheres S_r and S_R on which the rotations of the completely continuous vector field $\boldsymbol{I} - \tilde{\boldsymbol{A}}$ are respectively*

$$\gamma(S_r) = 0, \tag{3.20}$$

and

$$\gamma(S_R) = 1. \tag{3.21}$$

Theorem B. *Let the conditions of theorem II be satisfied, or let the conditions of theorem IV be satisfied and in addition let the cone K be normal.*† *Then there exist spheres S_r and S_R on which the rotations of the completely continuous vector field $\boldsymbol{I} - \tilde{\boldsymbol{A}}$ are respectively*

$$\gamma(S_r) = 1, \tag{3.22}$$

and

$$\gamma(S_R) = 0. \tag{3.23}$$

Proofs of these two theorems are given in appendix 1. We note that they are themselves in effect existence theorems. For, according to either *A* or *B*, the rotation of the field $\boldsymbol{I} - \tilde{\boldsymbol{A}}$ is different from zero on the boundary $S_r + S_R$ of the bounded domain $T_{r,R}$ which consists of the set of points u satisfying $r < \|u\| < R$. By virtue of the general Leray–Schauder principle (see § 2.3) it follows that at least one fixed point of $\tilde{\boldsymbol{A}}$ exists in $T_{r,R}$, and, for the reason noted earlier, this must also be a fixed point of $\boldsymbol{A}$.

Allowing that there may be several fixed points in $K \cap T_{r,R}$, let us label them ϕ_j $(j = 1, 2, \ldots, k)$ and denote their indices respectively by γ_j. As was noted in § 2.3, the number k is necessarily finite if the fixed points are isolated. If K is a solid cone and the non-zero fixed points are interior elements, so that $\tilde{\boldsymbol{A}}u = \boldsymbol{A}u$ on a sufficiently small sphere surrounding each ϕ_j, then clearly the index of ϕ_j considered as a solution of $\phi = \tilde{\boldsymbol{A}}\phi$ is the same as the index of this fixed point considered as a solution of $\phi = \boldsymbol{A}\phi$. This equivalence is not immediately obvious if the cone has no interior and we leave this case until later, proceeding for now on the assumption of the simpler case.

Implications of theorem A and theorem B

To establish the practical bearing of these theorems, appeal is made to the properties of the derivative operator $\boldsymbol{A}'(\phi_j)$ that were presupposed in the discussion at the end of § 3.1, where in turn reference was made to the concluding paragraph of § 2.1. Let λ_j denote for short the eigenvalue (previously written λ_ϕ with regard to a general fixed point) to which a positive eigenvector of $\boldsymbol{A}'(\phi_j)$ corresponds. We take each λ_j to be simple, unique and greater in magnitude than the remaining eigenvalues of $\boldsymbol{A}'(\phi_j)$, which will be denoted by $\lambda_j^{(m)}$ $(m = 2, 3, \ldots)$. The following facts are then apparent in the light of the formula (2.15) and its context:

(*a*) If $\lambda_j < 1$ (i.e. the fixed point ϕ_j represents a supercritical conjugate flow), then $\gamma_j = 1$.

† This additional condition is a natural one as regards the hydrodynamical problem. The normality of the cone would generally have to be used in establishing the first condition of theorem I or theorem II on the assumption that the primary flow is, respectively, subcritical or supercritical.

(*b*) If unity is not an eigenvalue of $A'(\phi_j)$, then ϕ_j is an isolated fixed point whose index satisfies $|\gamma_j| = 1$.

(*c*) Subject to the condition in (*b*), $\gamma_j = -1$ implies that $\lambda_j > 1$, so that the conjugate flow represented by ϕ_j is subcritical.

(*d*) The converse of (*a*) is not necessarily true. If $\gamma_j = 1$ it is possible, of course, that $\lambda_j < 1$; but another possibility is that there is an even number of eigenvalues of $A'(\phi_j)$ in excess of unity. Thus, for instance, $\lambda_j > \lambda_j^{(2)} > 1 > \lambda_j^{(m)}$ $(m = 3, 4, \ldots)$. Suppose, however, that eigenvalues other than λ_j cannot exceed unity, a fact that might be established by a study of the operator A in specific examples. Then $\gamma_j = 1$ implies that $\lambda_j < 1$, so that the conjugate flow represented by ϕ_j is supercritical.

For the reasons pointed out in §1, the exceptional case in which unity is an eigen value o $A'(\phi_j)$ might justifiably be excluded by assumption. The index of a fixed point may still be definable in this case, however, and the range of the present interpretation can be considerably extended by means of the proposition explained at the end of §2.4. We assume this proposition applies, and in particular the operator A has the very usual property noted in the final paragraph of §2.4 (i.e. so that $s = 2$). Thus we have $\gamma_j = 0$ in the exceptional case. A fixed point ϕ_j will be called *exceptional* or *unexceptional* accordingly as $\gamma_j = 0$ or, by (*b*) above, $|\gamma_j| = 1$.

The implications of theorem A and theorem B can now be explained, provisionally at least on the basis of the assumption that the cone K is solid and the non-zero fixed points ϕ_j are interior elements. Applying the topological principle expressed by (2.14) and then its particular form expressed by (2.13), we conclude that if the primary flow is subcritical and accordingly theorem A applies, then

$$\gamma_1 + \gamma_2 + \ldots + \gamma_k = 1 \tag{3.24}$$

by virtue of (3.20) and (3.21). Again, if the primary flow is supercritical and accordingly theorem B applies, then

$$\gamma_1 + \gamma_2 + \ldots + \gamma_k = -1 \tag{3.25}$$

by virtue of (3.22) and (3.23). In either case it follows that the number of unexceptional fixed points is odd, say $1 + 2N$ $(N \geqslant 0)$. And if, as proposed in (*d*) above, we assume that for each fixed point the eigenvalues of $A'(\phi_j)$ other than λ_j cannot exceed unity, then an especially simple interpretation of the transcritical property follows immediately. That is, in the case of a subcritical primary flow $1 + N$ of these fixed points represent supercritical conjugate flows, while N represent subcritical conjugate flows; and the corresponding statement with the words subcritical and supercritical interchanged is also true. Thus we have a complete counterpart to the set of conclusions that was pointed out in §1 with regard to the rudimentary example in which (3.1) is a scalar equation.

The general case of a supercritical primary flow

When the restriction on eigenvalues other than λ_j is not imposed, conclusions that are almost as orderly can be made. Suppose first that the primary flow is supercritical and theorem B applies. The result (3.25) shows that there are $1 + N \geqslant 1$ unexceptional fixed points for which $\gamma_j = -1$, so that, by (*c*) above, at least $1 + N$ subcritical conjugate flows exist. The main conclusion thus established deserves an emphatic statement as follows:

If the primary flow is supercritical (*in the sense explained in* §2.1) *and if conditions obtain* (*i.e. those of theorem B*) *that guarantee at least one non-trivial solution of equation* (3.1) *in the cone K, then a* SUBCRITICAL *conjugate flow exists.*

The general case of a subcritical primary flow

Suppose now that the primary flow is subcritical and theorem A applies. The result (3.24) shows that there are $1+N \geqslant 1$ unexceptional fixed points for which $\gamma_j = 1$; but in general, as noted in (*d*) above, these do not necessarily represent supercritical flows. To make progress with this case we rely in the first place on the assumption that the operator $\boldsymbol{A}$ is monotonic on K.

Let us begin by identifying an unexceptional fixed point, say ϕ_k, that satisfies

$$\phi^* - \phi_k \notin K, \tag{3.26}$$

where, as introduced in § 3.3, ϕ^* stands for each in turn of the other non-zero fixed points. It is easy to see that at least one among several fixed points always satisfies (3.26). We next define a new space E_k by transferring the zero point θ to the point ϕ_k in the original space E, and we consider the cone K_k in E_k consisting of the collection of elements from E for which

$$w = u - \phi_k \in K. \tag{3.27}$$

Since $\phi_k = \boldsymbol{A}\phi_k$, equation (3.1) can be written

$$w = \boldsymbol{A}(\phi_k + w) - \boldsymbol{A}\phi_k = \boldsymbol{A}_k w, \quad \text{say,} \tag{3.28}$$

and the fact that $\boldsymbol{A}$ is monotonic implies that the new operator $\boldsymbol{A}_k$ is positive on K_k. Also, of course, we have $\boldsymbol{A}_k\theta = \theta$. If we now suppose that the conjugate flow represented by ϕ_k is subcritical, we arrive at a contradiction. In fact, if $\lambda_k > 1$ the first condition of theorem III is provided with regard to the positive operator $\boldsymbol{A}_k$, and the application of theorem A leads easily to the conclusion that a non-zero fixed point of $\boldsymbol{A}_k$ exists in K_k. But this contradicts the condition (3.26) whereby ϕ_k is selected. Thus it appears that the conjugate flow in question must be supercritical ($\lambda_k < 1$).

This aspect of the transcritical property is complementary to the one established in § 3.3, where too the assumption that $\boldsymbol{A}$ is monotonic was required. [Without this or some other additional assumption, the conditions of theorem A seem insufficient to guarantee a supercritical conjugate flow, although the existence of such a flow is in fact the simplest outcome of variational methods if they happen to be applicable (see § 3.5).] If $\boldsymbol{A}$ is monotonic and if only a single unexceptional fixed point ϕ_1 exists, then from either the present or the former result it appears that the conjugate flow represented by ϕ_1 is supercritical. Combining the two results we may also conclude that if three unexceptional fixed points exist, the nearest to and the farthest from the zero point both represent supercritical flows, while the third represents a subcritical flow and is intervening in the sense of not satisfying either (3.14) or (3.26) [see figure 2 (*a*) introduced below]. Our main general conclusion in the present case is worth stating emphatically as follows:

If the primary flow is subcritical (*in the sense explained in* § 2.1) *and if conditions obtain* (*i.e. those of theorem A*) *that guarantee at least one non-zero positive solution of equation* (3.1), *in which the nonlinear operator* $\boldsymbol{A}$ *is monotonic on* K, *then a* SUBCRITICAL *conjugate flow exists.*

The preceding form of argument can be referred to any one of the unexceptional fixed points ϕ_j, leading to expressions for the sum of the indices of fixed points that lie inside the cone K_j radiating from ϕ_j. If ϕ_j is subcritical theorem A and its corollary (3.24) apply, as considered above; while theorem B and its corollary (3.25) apply if ϕ_j is supercritical. The conclusions are listed as follows, where γ_p stands for the indices of those unexceptional fixed points lying inside a particular K_j (i.e. such that $\phi_p - \phi_j \in K$, $p \neq j$), β is the number of eigenvalues of $A'(\phi_j)$ exceeding unity, and $q = \sum |\gamma_p|$ is the number of fixed points ϕ_p:

(i) $\gamma_j = 1$, $\beta = 0$, so ϕ_j is supercritical: then $\sum\gamma_p = 0$ (q even);

(ii) $\gamma_j = 1$, $\beta \geqslant 2$ (β even), so ϕ_j is subcritical: then $\sum\gamma_p = 1$ (q odd);

(iii) $\gamma_j = -1$, $\beta \geqslant 1$ (β odd), so ϕ_j is subcritical: then $\sum\gamma_p = 1$ (q odd).

These propositions are illustrated in figure 2, which shows possible dispositions of unexceptional fixed points of a monotonic positive transformation in R_2, the cone being the first quadrant of the plane. It may be seen that these diagrams are in accord with the foregoing conclusions, and with the one established in § 3.3. For three fixed points the case (ii) listed above is impossible: figure 2 (*a*) shows a possible arrangement of the two fixed points that must represent supercritical flows, and the remaining one that must represent a subcritical flow and so cannot satisfy either (3.14) or (3.26). Figures 2 (*b*) and (*c*) show five fixed points, and case (ii) is illustrated by the circled point in 2 (*c*). Figure 2 (*d*) shows seven fixed points, including two instances of case (ii).

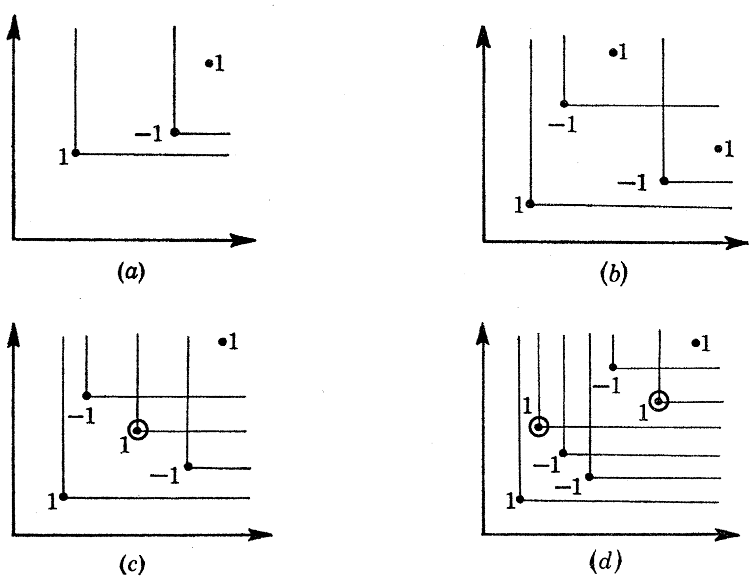

FIGURE 2. Possible dispositions in the first quadrant of fixed points of a positive transformation. The index of each fixed point is shown: $\gamma = -1$ implies a subcritical flow; $\gamma = 1$ implies a supercritical flow except for circled points.

Propositions corresponding to (i) to (iii) can similarly be established for the case where the primary flow is supercritical and theorem B is applicable. The sums $\sum\gamma_p$ have the values -1, 0 and 0, respectively, in the counterparts to (i), (ii) and (iii).

Uniqueness

If theorem A is applicable and it can be shown that $\gamma = 1$ for any fixed point $\phi \in K$ ($\phi \neq \theta$), then evidently the fixed point guaranteed by theorem A is unique. Again, a unique non-zero fixed point is established by theorem B if it can be shown that $\gamma = -1$. To illustrate the use of this topological proof of uniqueness, we consider the first case, in which the primary flow is subcritical and theorem A applies, and we proceed on the basis of assumptions like those made in § 3.2. That is, K is taken to be a cone of non-negative functions and the positive operator A to be *concave*, thus satisfying (3.9). It is further assumed that the Fréchet derivative $A'(\phi)$ in the directions of K is self-adjoint with respect to some positive weight function g [see (2.21)], and has an eigenvector $\eta \in K$.

The result (3.11) is now recalled from § 3.2. Thus, owing to the concavity of $\boldsymbol{A}$, a positive number $t_0 < 1$ can be chosen so that

$$\boldsymbol{A}(t_0\phi) \geqslant t_0(\phi+\chi) \quad \text{with} \quad \chi \in K \quad (\chi \neq \theta).$$

Using this result after putting $u_1 = t_0\phi$ and $u_2 = \phi$ in the definition (3.9) of concavity, we obtain

$$\begin{aligned} \boldsymbol{A}\{t't_0\phi+(1-t')\,\phi\} &\geqslant t'\boldsymbol{A}(t_0\phi)+(1-t')\,\boldsymbol{A}\phi \\ &\geqslant t't_0(\phi+\chi)+\boldsymbol{A}\phi-t'\phi, \end{aligned}$$

which leads upon rearrangement to

$$(1-t_0)\,\phi \geqslant \frac{\boldsymbol{A}\phi-\boldsymbol{A}\{\phi-t'(1-t_0)\,\phi\}}{t'}+t_0\chi \quad \text{for} \quad 0 \leqslant t' \leqslant 1.$$

In the limit as $t' \to 0$ this gives

$$\phi \geqslant \boldsymbol{A}'(\phi)\,\phi+b\chi, \tag{3.29}$$

where $b = t_0/(1-t_0) > 0$. Hence, introducing the eigenvector η of $\boldsymbol{A}'(\phi)$ for which the eigenvalue is λ_ϕ, assuming that this non-negative function is positive almost everywhere, and using the self-adjointness of $\boldsymbol{A}'(\phi)$ with respect to the positive weight function g, we have

$$\begin{aligned} 0 < \langle b\chi, \eta\rangle_g &= \langle\{\phi-\boldsymbol{A}'(\phi)\,\phi\}, \eta\rangle_g = \langle\phi, \eta\rangle_g-\langle\phi, \boldsymbol{A}'(\phi)\,\eta\rangle_g \\ &= (1-\lambda_\phi)\,\langle\phi, \eta\rangle_g, \end{aligned} \tag{3.30}$$

which shows that $\lambda_\phi < 1$. It follows from the fact (*a*) listed above that $\gamma = 1$, and we therefore conclude that the fixed point ensured by theorem A is unique.

If $\boldsymbol{A}$ is a convex operator the inequality $\lambda_\phi > 1$ can similarly be established. To prove uniqueness, however, it is necessary to show additionally that the remaining eigenvalues of $\boldsymbol{A}'(\phi)$ are less than unity [see fact (*d*) above]. This may be done by a study of the operator $\boldsymbol{A}$ in some examples [cf. Benjamin 1971, § 5, where a uniqueness theorem is proved by a different method, which nevertheless is inherently equivalent to the present approach]. We note that uniqueness theorems for fixed points of concave operators acting in a general cone have been given by Krasnosel'skii (K (*b*), p. 281 *et seq.*; K (*c*), § 6.1), although a different definition of concavity was proposed by him. With regard to problems of uniqueness it is well known that nonlinear equations involving convex operators pose special difficulties, not common to equations with concave operators, and the pressent topological considerations give an interesting sidelight on the matter.

Application to cones without interiors

It remains to show that the foregoing conclusions usually remain valid in cases where the cone K is not solid. In the last part of the following discussion we shall particularly consider the cone of non-negative functions in L_2, but the argument will be seen to apply equally well to cones of non-negative functions in other L_p spaces and to non-solid cones in C (see § 6.4). As it appears in (3.24) and (3.25), the index γ of a particular non-zero fixed point $\phi \in K$ is supposed to be the rotation of the completely continuous vector field $\boldsymbol{I}-\tilde{\boldsymbol{A}}$ on a small sphere surrounding the fixed point. But if K has no interior such a sphere can only intersect with and must lie partly outside K. So, because $\tilde{\boldsymbol{A}}$ is not necessarily identical with $\boldsymbol{A}$ outside K, it is not obvious that the rotation of the field $\boldsymbol{I}-\tilde{\boldsymbol{A}}$ should be the same as the rotation of $\boldsymbol{I}-\boldsymbol{A}$. Our object is to show that γ as here defined depends on the eigenvalues of $\boldsymbol{A}'(\phi)$ in the same way as the index of a fixed point of $\boldsymbol{I}-\boldsymbol{A}$.

Supposing first that the eigenvalues of $\boldsymbol{A}'(\phi)$ are all less than unity, we shall show that the completely continuous field consisting of vectors $u - \tilde{A}u$ $(u \in E)$ is homotopic to $u - \phi$ on a sufficiently small sphere surrounding the fixed point $\phi \in K$. Thus, for the rotation on this sphere, we have $\gamma = 1$ (see § 2.4). The homotopy is established if the completely continuous vector fields

$$F(u,t) = u - t\tilde{A}u - (1-t)\,\phi \quad (0 \leqslant t \leqslant 1)$$

can be shown to include no zero vector. Assuming to the contrary that there exists an element u_0 such that $F(u_0, t) = \theta$, we find at once that

$$u_0 = t\tilde{A}u_0 + (1-t)\,\phi \in K, \tag{3.31}$$

since $\tilde{A}u_0 \in K$ by the definition (3.18). Hence $\tilde{A}u_0 = Au_0$ by (3.19), and a rearrangement of (3.31) after the substitution of $\phi = A\phi$ gives

$$h = t\{A(\phi + h) - A\phi\}, \tag{3.32}$$

where $h = u_0 - \phi$. By adaptation of a standard argument which was cited in § 2.4 (see, for example, K (*b*), pp. 136, 137), it can easily be shown that if $\|h\|$ is sufficiently small, the existence of a solution of (3.32) implies that the linear equation

$$h = tA'(\phi)\,h \tag{3.33}$$

has a solution. Since $A'(\phi)$ is specified to have no eigenvalue equal to or greater than unity, whereas $1/t \geqslant 1$, we have thus arrived at a contradiction and so the proposed fact $\gamma = 1$ is proved.

The other case that needs to be considered is where $A'(\phi)$ has a number β of (simple) eigenvalue greater than unity and none equal to unity. The treatment of this more difficult case is abbreviated in several respects, since in large part standard arguments are followed. Let E_1 denote the β-dimensional subspace of E comprised from the set of eigenvectors of $A'(\phi)$ corresponding to eigenvalues greater than unity. Evidently E_1 is an invariant subspace for the linear operator $A'(\phi)$. According to a well-known theory originated by F. Riesz, the space E can be represented as the direct sum of E_1 and a subspace E_2 which includes no eigenvector of $A'(\phi)$ corresponding to an eigenvalue greater than unity. (In fact, E_2 is the closure of the direct sum of all invariant subspaces for $A'(\phi)$ which do not intersect E_1.) Putting as before $u = \phi + h$ with $h \in E$, we have to find the rotation γ of the completely continuous field $I - \tilde{A}$ on a sphere S_ρ, i.e. $\|h\| = \rho$, where ρ is sufficiently small. Let $S_1 = S_\rho \cap E_1$ and $S_2 = S_\rho \cap E_2$ denote the intersections of S_ρ with the complementary subspaces E_1 and E_2. A topological fact which is crucial here is that $\gamma = \gamma_1 . \gamma_2$, where γ_1 is the rotation of the field on S_1 and γ_2 is the rotation on S_2 (K (*b*), p. 129, theorem 4.5: see particularly note on p. 132). On S_2 there is no eigenvector of $A'(\phi)$ corresponding to an eigenvalue greater than unity, and by assumption no eigenvalue equals unity; hence the argument given in the preceding paragraph shows that if ρ is sufficiently small, the field $u - \tilde{A}u$ is homotopic to $u - \phi$ on S_2 and therefore $\gamma_2 = 1$. Thus we have $\gamma = \gamma_1$.

The remainder of the argument rests on the often applicable proposition that the finite-dimensional sphere S_1 is contained completely in the cone K, provided ρ is small enough. To fix ideas we take E specifically to be L_2, and we assume that the individual eigenvectors of $A'(\phi)$ are continuous functions—i.e. they can also be considered as elements of the space C. [Note that this is a natural assumption with regard to linear integral operators in L_2 (cf. § 6).] Let η_m $(m = 1, 2, \ldots, \beta)$ denote the eigenvectors of $A'(\phi)$ belonging to E_1; and for simplicity (although this is not essential) suppose that, as is usual, these eigenvectors are orthogonal with respect to an inner product of the type (2.21) with a weight function g that is positive and bounded. Also let them be normalized with respect to this inner product: that is, $\langle\eta_m, \eta_n\rangle_g = \delta_{mn}$. Then any element of E_1 has the form

$$h_1 = a_1\eta_1 + \ldots + a_\beta\eta_\beta, \tag{3.34}$$

and its L_2 norm, which equals ρ on S_1, satisfies

$$(\max g)^{\frac{1}{2}}\,\|h_1\|_{L_2} \geqslant [\langle h_1, h_1\rangle_g]^{\frac{1}{2}} = (a_1^2 + \ldots + a_\beta^2)^{\frac{1}{2}}. \tag{3.35}$$

As already explained, we assume that there exist finite positive numbers C_m $(m = 1, 2, \ldots, \beta)$ given by

$$C_m = \|\eta_m\|_C = \max|\eta_m|. \tag{3.36}$$

From (3.34) to (3.36) it follows that on S_1

$$\begin{aligned}\|h_1\|_C = \max|h_1| &\leqslant C_1|a_1| + \ldots + C_\beta|a_\beta| \\ &\leqslant (C_1 + \ldots + C_\beta)\,(\max g)^{\frac{1}{2}}\rho.\end{aligned} \tag{3.37}$$

Thus $\|h_1\|_C$ can be made arbitrarily small by taking ρ small enough.

Now suppose that ϕ has been established in the first place as a fixed point in the cone of non-negative functions in L_2, but also is an interior element of the cone of non-negative functions in C. It is then evident from (3.37) that provided ρ is sufficiently small, the finite-dimensional sphere S_1 centred on ϕ is contained in the latter cone and therefore also in the former. In applications of the present theory, however, cases are presented where the solution ϕ belongs to the cone of non-negative functions in C but is not an interior element: this situation was explained in the context of (2.1). The required conclusion may then be established by use of the norm given by (2.1), which defines a Banach space of continuous functions vanishing at the boundary of their domain of definition. The solution ϕ is an interior element of the solid cone of non-negative functions in this new space; and if, as is usual in such cases, the eigenvectors η_m are also elements of this space, then an argument corresponding to the above shows that again S_1 is contained within the cone of non-negative functions in L_2.

With this fact ascertained, we know that $\tilde{\boldsymbol{A}}u = \boldsymbol{A}u$ on S_1 and so the evaluation of γ_1 can proceed by the standard argument (K (b), pp. 133 to 137). If ρ is sufficiently small, the field consisting of vectors $\phi + h_1 - \boldsymbol{A}(\phi + h_1)$ $(h_1 \in S_1)$ is homotopic to $\boldsymbol{I} - \boldsymbol{A}'(\phi)$, which in turn is homotopic to the field $-\boldsymbol{I}$ on S_1. The latter result is proved by considering the fields

$$(2t-1)\,h_1 - t\boldsymbol{A}'(\phi)\,h_1 \quad (0 \leqslant t \leqslant 1),$$

the vanishing of which anywhere on S_1 would imply the existence of an eigenvector of $\boldsymbol{A}'(\phi)$ corresponding to an eigenvalue less than or equal to unity, contrary to the definition of the subspace E_1. On a sphere surrounding the origin in a β-dimensional space, the rotation of the field $-\boldsymbol{I}$ (i.e. the field of interior normals) is equal to $(-1)^\beta$ (K (b), p. 92; Aleksandrov 1960, p. 112, proposition 5.14). Hence from the stated homotopy it follows that $\gamma_1 = (-1)^\beta$, and we finally conclude that $\gamma = (-1)^\beta$, which is the anticipated result.

3.5. *Flow force*

We here examine some useful concepts that appear if the governing equation can be recast in, or perhaps originally has, a form derivable from a variational principle. Our attention focuses particularly on the reformulated equation $\zeta = \boldsymbol{G}\zeta$, whose solution ζ representing a conjugate flow can be regarded as a null point of a gradient field in a Hilbert space E. Thus $\boldsymbol{G}$ is a potential operator (see § 2.5), and we have

$$\zeta - \boldsymbol{G}\zeta = \operatorname{grad} \Lambda(\zeta) = \theta, \tag{3.38}$$

where the functional (scalar) Λ has the form

$$\Lambda(u) = \tfrac{1}{2}\langle u, u\rangle - \Omega(u), \tag{3.39}$$

already noted as (2.23). In various examples of conjugate flows in frictionless systems, it turns out that $\Lambda(\zeta)$ is actually equal to (or at least proportional to) the difference in flow force between the flow represented by ζ and the primary flow. (We recall that flow force is defined as the sum of pressure force and momentum flux acting in the x-direction through a cross-section of the flow.) More generally, if an expression for flow force is varied subject to conditions of mass and energy conservation, it appears that conjugate flows have the property of making this expression stationary. This principle is well known with regard to open-channel flows, and it was used by Benjamin (1962) in treating the more difficult—and formally quite different—problem of conjugate vortex flows. A detailed example of the principle will be worked out in § 6.5.

It is of considerable interest to ask why this remarkable property of frictionless conjugate flows arises. That is, given that the equation determining steady x-independent flows has the form (3.38), why is it that $\Lambda(\zeta)$ usually represents relative flow force? This question evidently turns on the basic hydrodynamics of frictionless flows, however, rather than on mathematical aspects of the operator equation to which the physical problem is reduced, and so it will be discussed separately in appendix 2. In the present approach we consider Λ *qua* the functional in (3.38), establishing on this basis certain facts about its possible values. These facts become especially significant in applications to frictionless flows where Λ can be identified with relative flow force, but we appreciate that there may be other applications, concerning viscous fluids, where Λ will mean something different. A particular aim is to illuminate the principle, already known from several examples, that in a transcritical pair of conjugate frictionless flows the supercritical flow has the smaller value of flow force.

The case of R_n

We interrupt the main discussion to note the somewhat different situation usually presented when the solution to a conjugate-flow problem is finite-dimensional (see examples in §§ 4 and 5). Suppose that in the original formulation of the problem, giving equation (3.1), the solution ϕ is an n-dimensional vector with components ϕ_i $(i = 1, 2, \ldots, n)$, to be considered as a column matrix. As was previously indicated by (1.11), equation (3.1) then stands for n simultaneous equations

$$\phi_i = A_i(\phi), \tag{3.40}$$

in which the nonlinear functions A_i are generally all different. The general question whether this system of equations can be reduced to a variational principle is evidently tied in with Pfaff's problem concerning differential forms. For instance, the possibility of the matrix transformation being itself the gradient of a scalar Ω, i.e.

$$A_i(\phi) = \partial\Omega(\phi)/\partial\phi_i,$$

is equivalent to the differential form

$$\sum_{i=1}^{n} A_i(\phi)\,\mathrm{d}\phi_i$$

being an exact differential. If $\mathbf{A}$ happens to be a potential transformation, then the main points of the following arguments apply immediately and the interpretation is simple. They would also obviously apply if, as will be assumed below with regard to L_2, a reduction of the problem to the form (3.38) is possible by means of a linear transformation $\phi = \mathbf{L}\zeta$.

In real examples, however, the more usual situation appears to be that a variational statement of the problem is obtainable by introducing an 'integrating factor': this is, we recall, the simplest device whereby a differential form might be converted into an exact differential. Thus one has in this case

$$Q(\phi)\{\phi_i - A_i(\phi)\} = \partial \bar{\Lambda}(\phi)/\partial\phi_i, \tag{3.41}$$

where the function $Q(\phi)$ is, of course, common to every equation corresponding to $i = 1, 2, \ldots, n$. We use the symbol $\bar{\Lambda}$ for the scalar in (3.41) to emphasize its analogy with Λ in (3.38), despite the formal difference due to the presence of the factor $Q(\phi)$. It may easily be seen as follows that, if $Q(\phi)$ is a positive function (which appears generally to be true in applications), the main conclusions reached below regarding the values of $\Lambda(\zeta)$ also apply to $\bar{\Lambda}(\phi)$ in the present case.

The subsequent argument essentially concerns the sign of the second variation of Λ at a solution point, considering particularly that the possibility of an invariable sign depends on the eigenvalues of the derivative operator at this point. In the finite-dimensional forms of the general problem, the definition of abstract differentiation given in § 2.1 shows that the derivative operator $\boldsymbol{A}'(\phi)$ is an n-dimensional square matrix with elements $\partial A_i(\phi)/\partial\phi_j$, in which i varies down the columns and j varies across the rows. And, in the case when (3.41) holds, the second variation of $\bar{\Lambda}$ at an arbitrary point ϕ [i.e. the second term of the expansion of $\Lambda(\phi+th)-\Lambda(\phi)$ in powers of t] is given by

$$\tfrac{1}{2}t^2 Q(\phi)\left\{h_i^2 - \frac{\partial A_i(\phi)}{\partial\phi_j}h_i h_j\right\} + \tfrac{1}{2}t^2\frac{\partial Q(\phi)}{\partial\phi_j}h_j\{\phi_i - A_i(\phi)\}h_i, \tag{3.42}$$

where repeated subscripts imply summation. The second group of terms in (3.42) vanishes at a point satisfying (3.40), and thus the possibility of an invariable sign is the same as if the positive function $Q(\phi)$ were a constant. In fact we see that the second variation of $\bar{\Lambda}$ at a solution point ϕ is not positive definite if the matrix $\boldsymbol{A}'(\phi)$ has an eigenvalue greater than unity. It will be appreciated from the subsequent discussion that this conclusion completely establishes the analogy with the problem posed in L_2.

We need to remember here that the operator $\boldsymbol{A}$ depends on the velocity parameter μ. In examples where the solution is a finite-dimensional vector the most convenient formulation of the problem may be such that $\boldsymbol{A}(\mu)$ is decreasing with μ, rather than increasing as was assumed for the sake of clarity in this part of the paper [see context of (3.1) and footnote]. The effect of velocity changes on the eigenvalues of $\boldsymbol{A}'(\phi)$ is then reversed in direction; and so if the scalar $\bar{\Lambda}$ is presented as having the property (3.41), it is to be expected that $-\bar{\Lambda}(\phi)$ rather than $\bar{\Lambda}(\phi)$ gives the flow force of the conjugate flow relative to the primary flow. A very simple example of this is shown in § 4.

The case of L_2

We return to the problem expressed by (3.38), considering particularly that the space E in which the potential operator $\boldsymbol{G}$ acts is L_2. It is assumed that $\boldsymbol{G}$ has a strong Fréchet derivative in every direction of E. The reduction of (3.1) to the form (3.38) is not, of course, necessarily possible if $\boldsymbol{A}$ is allowed to be a general nonlinear operator, but the case where it can be so reduced appears typical of the theory of frictionless conjugate flows. Following Krasnosel'skii (K (*a*), p. 349), we first note that the reduction is simple if $\boldsymbol{A}$ happens to be representable in the form $\boldsymbol{A} = \boldsymbol{LM}$, where the operators $\boldsymbol{L}$ and $\boldsymbol{M}$ are such that $\boldsymbol{ML}$ is potential and moreover $\boldsymbol{L}u = \theta$ only if $u = \theta$. For then the substitution of $\phi = \boldsymbol{L}\zeta$ in (3.1) leads to the equation $\zeta = \boldsymbol{ML}\zeta$, which has the required form. The outstanding instance of this device is in the well-known variational method for treating equations of the Hammerstein type (K (*a*), p. 363; K (*b*), p. 340; Vainberg 1964, ch. 7), and it is this type of nonlinear integral equation that is chiefly met in applications of the present theory (see appendix 2 for reasons). Referring for details to the example treated in § 6 and to appendix 2, we merely note here that, for the case in question, the operator in (3.1) takes the form $\boldsymbol{A} = \boldsymbol{BF}$, where $\boldsymbol{B}$ is a self-adjoint linear integral operator having only positive eigenvalues and $\boldsymbol{F}$ is a non-linear continuous operator of the sort exemplified in equation (1.4). The fact that $\boldsymbol{B}$ has no negative eigenvalue implies that a self-adjoint operator $\boldsymbol{B}^{\frac{1}{2}}$ exists such that $\boldsymbol{B}^{\frac{1}{2}}(\boldsymbol{B}^{\frac{1}{2}}u) = \boldsymbol{B}u$ for $u \in L_2$. Hence the substitution of $\phi = \boldsymbol{B}^{\frac{1}{2}}\zeta$ in (3.1) leads to $\zeta = \boldsymbol{B}^{\frac{1}{2}}\boldsymbol{F}(\boldsymbol{B}^{\frac{1}{2}}\zeta)$, in which $\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{F}\boldsymbol{B}^{\frac{1}{2}}$ is found to be a potential operator (i.e. $\boldsymbol{L} = \boldsymbol{B}^{\frac{1}{2}}$ and $\boldsymbol{M} = \boldsymbol{B}^{\frac{1}{2}}\boldsymbol{F}$ in the preceding notation).

As a basis for proceeding with the discussion of the abstract problem, we make an assumption which includes the important case just described but which appears slightly more general. The assumption is that the reduction of (3.1) to the form (3.38) is effected by some *linear* transformation

$$\phi = \boldsymbol{L}\zeta, \tag{3.43}$$

which is self-adjoint in L_2 and such that $\boldsymbol{L}u = \theta$ only if $u = \theta$. Thus (3.1) gives

$$\zeta = \boldsymbol{L}^{-1}\boldsymbol{A}(\boldsymbol{L}\zeta) = \boldsymbol{G}\zeta, \tag{3.44}$$

in which $\boldsymbol{G} = \boldsymbol{L}^{-1}\boldsymbol{A}\boldsymbol{L}$ is supposed to be a potential operator. The inverse $\boldsymbol{L}^{-1}$ will generally be an unbounded operator, but the composite operator in (3.44) may well be completely continuous in L_2 if $\boldsymbol{A}$ is completely continuous (e.g. this is true in the application to Hammerstein operators). Since $\boldsymbol{L}^{-1}\boldsymbol{L} = \boldsymbol{I}$ and $\boldsymbol{L}$ is self-adjoint, it is evident that $\boldsymbol{L}^{-1}$ is defined and self-adjoint on the subset $\boldsymbol{L}(E)$ of E, i.e. the subset into which the whole space E is mapped by $\boldsymbol{L}$.

The crucial fact implied by the present assumption is that the eigenvalues of the derivative operator $\boldsymbol{G}'(\zeta)$ at a solution point ζ for (3.38) [i.e. at a stationary point of the functional Λ] are the same as the eigenvalues of $\boldsymbol{A}'(\phi)$ at a solution point ϕ for (3.1). From (3.44) it follows that

$$\boldsymbol{G}'(\zeta) = \boldsymbol{L}^{-1}\boldsymbol{A}'(\boldsymbol{L}\zeta)\,\boldsymbol{L} = \boldsymbol{L}^{-1}\boldsymbol{A}'(\phi)\,\boldsymbol{L},$$

and so the equation satisfied by an eigenvector ϖ of $\boldsymbol{G}'(\zeta)$ is

$$\lambda\varpi = \boldsymbol{L}^{-1}\boldsymbol{A}'(\phi)\,\boldsymbol{L}\varpi.$$

Hence the substitution of $\eta = \boldsymbol{L}\varpi$ leads to

$$\lambda\eta = \boldsymbol{A}'(\phi)\,\eta,$$

showing that λ is simultaneously an eigenvalue of $\boldsymbol{G}'(\zeta)$ and of $\boldsymbol{A}'(\phi)$.

Now, $\Lambda(\zeta)$ is by definition a stationary value of the functional $\Lambda(u)$, but it need not be a minimum. In fact, a necessary condition for it to be a minimum is that the second variation of Λ at the point ζ should not be negative. On the assumption that the operator $\boldsymbol{G} = \operatorname{grad}\Omega$ is strongly differentiable, the second variation of Λ, denoted by $\delta^2\Lambda(\zeta)$, may be understood as $\frac{1}{2}t^2$ times the second differential of Λ in an arbitrary direction $th \in E$; and so it appears that

$$\delta^2\Lambda(\zeta) = \tfrac{1}{2}t^2\langle h, \{h - \boldsymbol{G}'(\zeta)\,h\}\rangle. \tag{3.45}$$

Thus we see that $\Lambda(\zeta)$ cannot be a minimum if the operator $\boldsymbol{G}'(\zeta)$ has an eigenvalue greater than unity. For, taking ϖ_1 to denote the eigenvector corresponding to the highest eigenvalue $\lambda_1 > 1$, we obtain a negative value of $\delta^2\Lambda(\zeta)$ on the substitution of $h = \varpi_1$ in (3.45). In the exceptional case when $\lambda_1 = 1$ the present argument is inconclusive; but a simple study based on the very general assumptions explained at the end of § 2.4 shows that, virtually always, $\Lambda(\zeta)$ is not a minimum in this case also. Hence the fact that λ_1 is also an eigenvalue of $\boldsymbol{A}'(\phi)$ leads us at once to an important conclusion in the light of the ideas developed in §§ 3.1 and 3.4. Identifying $\Lambda(\zeta)$ with flow force, we state the conclusion in physical terms as follows:

THE PRINCIPLE OF MINIMUM FLOW FORCE. *A flow that realizes a minimum value of flow force is necessarily supercritical.*

The converse statement can also be justified under present assumptions. Since $\boldsymbol{G}'(\zeta)$ is taken to be a strong Fréchet derivative, the remainder $\Lambda(\zeta+th)-\Lambda(\zeta)-\delta^2\Lambda(\zeta)$ is $o(t^2\|h\|^2)$. And if the highest eigenvalue of $\boldsymbol{G}'(\zeta)$ is less than unity (which is what we mean by the respective flow being supercritical), it follows that $\delta^2\Lambda(\zeta) \geqslant \alpha t^2\|h\|^2$, where α is a positive constant. Hence $\Lambda(\zeta+th)-\Lambda(\zeta)$ is positive in a sufficiently small neighbourhood of the point ζ. Thus in physical terms the conclusion may be stated: *A supercritical flow realizes a minimum value of flow force.*

Flow force of a supercritical conjugate flow

The principle of minimum flow force becomes particularly relevant when variational methods can be used to prove the existence of a conjugate flow corresponding to a subcritical primary flow (see § 6.5). The essentials of the argument that may be used in this case were noted in § 2.5. We recall that if the operator $\boldsymbol{G}$ is completely continuous, the functional $\Lambda(u)$ is weakly lower semi-continuous and therefore it will achieve its minimum value on any bounded, weakly closed subset σ of E. In the present application the suitable choice of σ is the closed ball defined by $\|u\| \leqslant R$, where the constant R is sufficiently large. We then have $\theta \in \sigma$, but the specification that the primary flow is subcritical precludes $\Lambda(\theta) = 0$ from being a minimum. The remaining step of the argument consists in showing that the minimum of Λ is not achieved on the spherical boundary of σ. Here appeal must be made to an assumption about the behaviour of the nonlinear operator for elements with large norms, just as was required in the topological arguments considered previously. If, for instance, it can be shown in this way that

$$\Omega(u) < \tfrac{1}{2}R^2 \quad \text{for} \quad \|u\| = R, \tag{3.46}$$

so that $\Lambda(u)$ has only positive values for $\|u\| = R$, then the proof is evidently complete. The conclusion is that a non-zero point $\zeta\,(\|\zeta\| < R)$ exists to which a minimum value $\Lambda(\zeta) < 0$ corresponds, and which is therefore a solution of (3.38). The general principle under discussion then establishes that the conjugate flow represented by ζ (and so having smaller flow force than the primary flow) must be supercritical.

[In specific applications it is often the case that, corresponding to the solution ζ which minimizes Λ, the solution $\phi = \boldsymbol{L}\zeta$ of the original equation (3.1) is a non-negative function. This fact may perhaps appear directly from a study of the operators $\boldsymbol{A}$ and $\boldsymbol{G}$: for instance, $\Lambda(\zeta)$ is evidently equivalent to $\tfrac{1}{2}\langle\zeta, \boldsymbol{G}\zeta\rangle-\Omega(\zeta)$, which may appear not to realize its least possible value if $\boldsymbol{L}\zeta$ has both positive and negative values (see the example of concave Hammerstein operators discussed at the end of this subsection). In this case the behaviour of the operators for functions u such that $\boldsymbol{L}u$ has negative values is essentially irrelevant, and indeed it may impede the establishment of a condition such as (3.46) needed to prove the existence of a solution. The preferable course then is to modify the operators so that only their relevant behaviour is entailed (e.g. the behaviour of $\boldsymbol{A}$ on a cone of non-negative functions): an example of this will be shown in § 6.5. Since the supercritical classification depends only on properties of the operators that will be preserved in such a modification, the principle of minimum flow force will be unaffected.]

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Flow force of a subcritical conjugate flow

Examples in which the primary flow is supercritical obviously require different treatment, because $\Lambda(\theta) = 0$ is then a minimum. Several approaches appear possible. One having helpful precedents (cf. Nehari 1961) is to restrict the functions in competition for a minimum of Λ by means of the normalization condition $\langle u, \boldsymbol{G}u\rangle \div \langle u, u\rangle = 1$. This condition is automatically satisfied by a non-zero solution of (3.38), so it is not a supplementary condition as in an isoperimetrical problem; but it is found to exclude a finite neighbourhood of the point θ. A more revealing approach, however, is provided by the ideas outlined as follows.

Assuming that the weakly continuous and differentiable functional $\Omega(u)$ has no stationary point other than θ [i.e. $\|\boldsymbol{G}u\| > 0$ if $\|u\| > 0$], and that Ω can have positive values on any sphere $\|u\| = \rho > 0$, we consider the maximum of Ω on each particular sphere. By virtue of the weak continuity of Ω in the Hilbert space E, this maximum exists, being assumed at some point u_* on the sphere (cf. Vainberg 1964, theorem 13.3, part 2). And according to a well-known principle the gradient of Ω must be collinear with the normal vector at the point of maximum, thus

$$\boldsymbol{G}u_* = \kappa u_*, \tag{3.47}$$

where κ ($\neq 0$) is a constant. We may describe u_* as an eigenvector of the nonlinear operator $\boldsymbol{G}$ corresponding to the eigenvalue κ. Since one such eigenvector exists on every sphere $\rho > 0$, the collection of them forms a continuous branch $B \subset E$, and the differentiability of $\Omega(u)$ and $\|u\|^2$ further implies that the spectrum of eigenvalues κ is continuous (cf. Vainberg 1964, theorem 13.11; K (*b*), p. 343, theorem 3.1). Therefore, to prove that equation (3.44) has a non-trivial solution, it is sufficient to show that the spectrum spans the value unity.

The specification of a supercritical primary flow means that $\boldsymbol{G}'(\theta)$ has no eigenvalue greater than or equal to unity, and from this it follows that $\kappa < 1$ if ρ is sufficiently small. As in the case considered previously, the final property to be established must depend on some assumption about the behaviour of the nonlinear operator for elements with large norms. Suppose that the element u_* realizing the maximum of $\Omega(u)$ subject to $\|u\| = \rho$ is necessarily such that $v_* = \boldsymbol{L}u_* \in K$, where K is a cone of non-negative functions in E (this is easily seen to be true in the case of convex Hammerstein operators which is discussed below). Suppose also that the original operator $\boldsymbol{A}$, related to $\boldsymbol{G}$ as shown in (3.44), 'expands' the large-norm part of the cone, i.e.

$$v - \boldsymbol{A}v \notin K \quad \text{if} \quad v \in K, \ \|v\| \geqslant R. \tag{3.48}$$

By use of the fact that $\Omega(u_*) \geqslant \Omega(u_0)$, where u_0 is an arbitrary element with $\|u_0\| = \rho$, it may generally be shown that $\|u_*\| \to \infty$ implies $\|v_*\| \to \infty$. Hence, in view of the result

$$\boldsymbol{A}v_* = \kappa v_*, \tag{3.49}$$

which is equivalent to (3.47), the condition (3.48) implies that $\kappa > 1$ if ρ is sufficiently large.

We can therefore conclude that an eigenvector ζ exists corresponding to $\kappa = 1$, with $\|\zeta\| = \rho_1$ say, and this represents a conjugate flow. It is possible that κ passes through the value unity more than once along the branch B of eigenvectors, in which case we specify ρ_1 to be the smallest ρ for which $\kappa = 1$. Properties of the conjugate flow may now be inferred by considering the flow-force functional Λ to be evaluated along B, so that the value taken on each sphere is evidently the minimum for that sphere. The normal derivative of Λ at the point u_* on the sphere $\|u\| = \rho$ is given by

$$\Lambda_n(u_*) = \frac{\langle u_*, (u_* - \boldsymbol{G}u_*)\rangle}{\rho} = (1-\kappa)\rho, \tag{3.50}$$

being therefore positive if $\rho < \rho_1$. And the derivative of Λ in the outward direction along B must have the sign as Λ_n, because in all directions orthogonal to the normal the derivative vanishes. Thus it appears that Λ increases along B up to the point ζ, and consequently $\Lambda(\zeta) > \Lambda(\theta) = 0$.

Clearly $\Lambda(\zeta)$ is not a minimum, being in fact maximal among the set of values $\Lambda(u_*)$ taken on B, and thus the solution ζ of (3.44) cannot be supercritical. Hence, excluding the exceptional case in which unity is an eigenvalue of $\boldsymbol{G}'(\zeta)$, we may infer that the solution is subcritical. It is a straightforward matter to show that one eigenvalue of $\boldsymbol{G}'(\zeta)$ exceeds unity, but this will be passed over.

Put in physical terms, the conclusion is that a conjugate flow exists which is subcritical and has a greater value of flow force than the supercritical primary flow.

Concave and convex operators

Finally we note a simple variational argument which is applicable when the solution ϕ of (3.1) lies in a cone K of non-negative functions and $\boldsymbol{A}$ is either a concave or convex *Hammerstein* operator, in the category that can be reduced to potential form. In the present notation the latter statement means that $\boldsymbol{A} = \boldsymbol{L}^2\boldsymbol{F}$ and $\boldsymbol{G} = \boldsymbol{LFL}$, and we assume that both the linear (completely continuous) operator $\boldsymbol{L}^2$ and the nonlinear continuous operator $\boldsymbol{F}$ (satisfying $\boldsymbol{F}\theta = \theta$) are positive on K. This implies that $\boldsymbol{F}$ is concave or convex on K if $\boldsymbol{A}$ has the respective property. [It is common in applications that the function $\boldsymbol{F}u$ has the form $u(y)f\{y, u(y)\}$, where $f(y, u)$ is a positive function: then $\boldsymbol{F}$ is concave or convex accordingly as $f(y, u)$ is a monotonic decreasing or monotonic increasing function of u.]

From the definition $\boldsymbol{G} = \operatorname{grad}\Omega$ it follows in this case that

$$\begin{aligned} \mathrm{d}\Omega(t\zeta)/\mathrm{d}t &= \langle \boldsymbol{G}(t\zeta), \zeta\rangle \\ &= \langle \boldsymbol{F}(t\boldsymbol{L}\zeta), \boldsymbol{L}\zeta\rangle = \langle \boldsymbol{F}(t\phi), \phi\rangle. \end{aligned}$$

Hence, using the fact that $\Omega(\theta) = 0$, we obtain

$$\Omega(\zeta) = \int_0^1 \langle \boldsymbol{F}(t\phi), \phi\rangle\,\mathrm{d}t. \tag{3.51}$$

Also, since ζ is a solution of (3.38), the definition (3.39) of $\Lambda(u)$ shows that

$$\Lambda(\zeta) = \tfrac{1}{2}\langle \boldsymbol{G}\zeta, \zeta\rangle - \Omega(\zeta) = \tfrac{1}{2}\langle \boldsymbol{F}\phi, \phi\rangle - \Omega(\zeta). \tag{3.52}$$

A combination of these two results gives

$$\Lambda(\zeta) = \int_0^1 \langle\{t\boldsymbol{F}\phi - \boldsymbol{F}(t\phi)\}, \phi\rangle\,\mathrm{d}t. \tag{3.53}$$

It is assumed that ϕ is a non-zero element of a cone K of non-negative functions. And if $\boldsymbol{F}$ is a concave operator according to the definition given in § 3.2, we have that $\boldsymbol{F}(t\phi) - t\boldsymbol{F}\phi$ is a non-zero element of K for $0 < t < 1$. Hence (3.53) shows immediately that

$$\Lambda(\zeta) < 0 \quad \text{if } \boldsymbol{F} \text{ is concave.} \tag{3.54}$$

It similarly appears that

$$\Lambda(\zeta) > 0 \quad \text{if } \boldsymbol{F} \text{ is convex.} \tag{3.55}$$

Inasmuch as they must exemplify the principle of minimum flow force, these conclusions are entirely consistent with the results obtained in §§ 3.2 and 3.4 concerning non-negative solutions of (3.1) when the operator is concave or convex. In the first case it was shown in § 3.2 that the primary flow must be subcritical, and (3.54) shows that the supercritical conjugate flow—which

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is unique if the arguments given in § 3.4 apply—has a smaller value of flow force. In the second case the primary flow must be supercritical, and so according to (3.55) the general principle is again borne out.

4. Example 1. Open-channel flow

We consider the flow of a heavy ideal fluid along a straight horizontal channel (see figure 3). The flow velocity is assumed to be uniform, so that the free surface is horizontal, and the greatest depth h of the stream is taken as the dependent variable. The cross-sectional area Σ of the stream is a monotonic increasing function of this variable, thus

$$\Sigma(h) = \int_0^h b(y)\,\mathrm{d}y,$$

and we shall use the fact that

$$\mathrm{d}\Sigma/\mathrm{d}h = b(h), \tag{4.1}$$

where $b(h)$ is the breadth of the channel at the free surface.

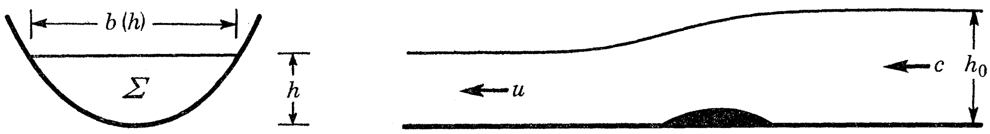

Figure 3. Illustration of open-channel flow.

Let h_0 denote the depth of the primary flow and c its (positive) velocity. The depth h and velocity u of any conjugate flow are related to h_0 and c by the condition of mass conservation

$$u\Sigma(h) = c\Sigma(h_0), \tag{4.2}$$

and the condition of energy conservation

$$gh + \tfrac{1}{2}u^2 = gh_0 + \tfrac{1}{2}c^2. \tag{4.3}$$

Hence, eliminating u and putting $h = h_0 + \phi$, we obtain

$$\phi = \frac{1}{2\mu}\left\{1 - \frac{\Sigma^2(h_0)}{\Sigma^2(h_0+\phi)}\right\} = \frac{\hat{A}(\phi)}{\mu}, \tag{4.4}$$

where $\mu = g/c^2$. (Note the inverse dependence on μ: see the first paragraph of § 3.1.)

From (4.4) and (4.1) there follows

$$\hat{A}'(0) = b(h_0)/\Sigma(h_0) = \mu_c, \quad \text{say.} \tag{4.5}$$

So, by our general criterion, the primary flow is supercritical or subcritical accordingly as $\mu < \mu_c$ or $\mu > \mu_c$. It is easily seen that the function

$$C(h) = \{g\Sigma(h)/b(h)\}^{\frac{1}{2}} \tag{4.6}$$

expresses the velocity of infinitesimal long waves relative to the fluid when the undisturbed depth is h. Thus the condition of supercritical flow is $c > C(h_0)$, in accord with the customary definition.

Let us suppose that Σ increases continuously with h and nowhere more rapidly than an exponential, so that $\Sigma\Sigma'' \leqslant (\Sigma')^2$ if Σ is twice differentiable. This implies that $\hat{A}(z)$ has a concave nonlinearity, i.e. $\hat{A}(tz) > t\hat{A}(z)$ for $0 < t < 1$; and hence the uniqueness and main properties of conjugate flows follow immediately by the general arguments of § 3. The conclusion is directly obvious from the graph of $\hat{A}(z)$ as drawn in figure 4, which includes the straight line μz for a supercritical and a subcritical value of μ. Since the curve $\hat{A}(z)$ has negative curvature between asymptotes $\frac{1}{2}$ for $z \to \infty$ and $-\infty$ for $\phi \to -h_0$, it appears that a unique conjugate solution $\phi \neq 0$ always exists. If the primary flow is supercritical ($\mu < \mu_c$), then the solution of (4.4) is positive and so $\mu^{-1}\hat{A}'(\phi) < 1$, which means that the conjugate flow is subcritical. And if the primary flow is subcritical ($\mu > \mu_c$), then the solution is negative and so $\mu^{-1}A'(\phi) > 1$, which means that the conjugate flow is supercritical.

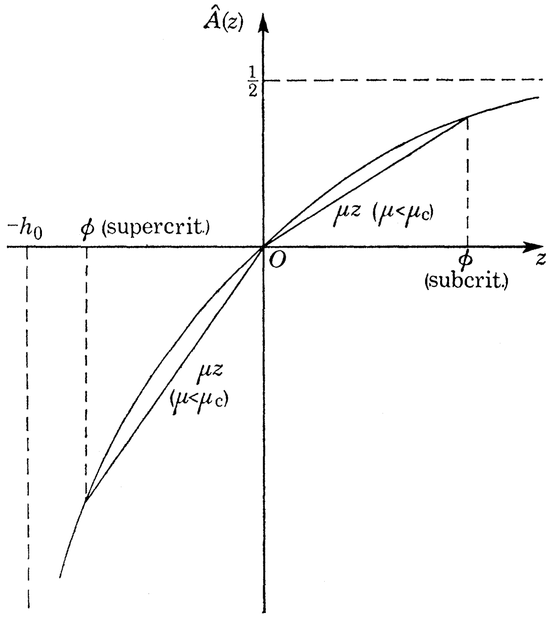

FIGURE 4. Graph of the function $\hat{A}(z)$ including the straight line μz for a supercritical and a subcritical value of μ.

The flow force S is defined as the sum of horizontal momentum flux and pressure force through a cross-section. Since the hydrostatic law of pressure applies, we have

$$S = \rho u^2 \Sigma + \int_0^h \rho g (h-y)\, b(y)\, \mathrm{d}y, \tag{4.7}$$

where ρ is the density of the fluid. In view of (4.2) and (4.3) this is equivalent to

$$\begin{aligned} S/\rho &= \tfrac{1}{2}u^2\Sigma + \int_0^h g(h-y)\, b(y)\, \mathrm{d}y + (-gh + \tfrac{1}{2}c^2 + gh_0)\, \Sigma \\ &= \frac{c^2\Sigma^2(h_0)}{2\Sigma(h)} + \int_0^h g(h-y)\, b(y)\, \mathrm{d}y + (-gh + \tfrac{1}{2}c^2 + gh_0)\, \Sigma. \end{aligned} \tag{4.8}$$

Hence we obtain, after using (4.1) and finding that two terms cancel,

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}h} &= \rho b(h) \left\{ -\frac{c^2\Sigma^2(h_0)}{2\Sigma^2(h)} - gh + \tfrac{1}{2}c^2 + gh_0 \right\} \\ &= \rho c^2 b(h_0+z)\, \{\hat{A}(z) - \mu z\}, \quad (h = h_0 + z). \end{aligned} \tag{4.9}$$

This result exemplifies the general principle noted in § 3.5: namely, when an expression for flow force is varied subject to conditions of mass and energy conservation, it takes stationary values for conjugate flows (i.e. for $z = \phi$, where ϕ is a solution of (4.4)).

Reference to figure 4 shows that, provided $\mu \neq \mu_c$, $\hat{A}(z) > \mu z$ if z lies in the open interval between the supercritical and subcritical solutions of (4.4), the latter of which always has a higher value than the former. It follows from (4.9), therefore, that a subcritical flow always has a larger value of flow force than its supercritical conjugate.

5. Example 2. Layered fluids

Systems comprising several fluids of different densities in superposed horizontal layers afford examples of conjugate flows for which the governing equation (3.1) takes a matrix form. If the composite fluid has $n+1$ discrete layers and is bounded at top and bottom by fixed horizontal planes, there are n free interfaces and the solution ϕ_i $(i = 1, 2, \ldots, n)$ representing the vertical displacements of these interfaces may be considered as a vector in a space of n dimensions. In specific cases a great many conjugate-flow pairs may be possible, so that a provisional ordering of the possibilities is generally essential as a first step towards obtaining comprehensible results. Various applications of the abstract theory can be found, depending on a suitable choice of cone in which the considered solution will lie. The following simple example suffices to demonstrate principles.

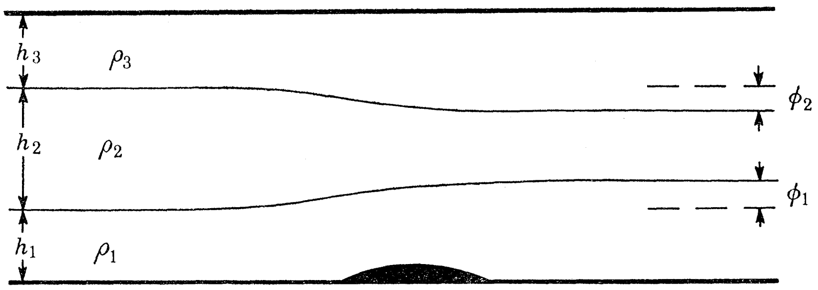

Figure 5. Illustration of three-layer system.

We consider the three-layer model illustrated in figure 5. In the primary state of the system, the depths of the lower, intermediate and upper layers are h_1, h_2 and h_3, and the respective fluid densities satisfy $\rho_1 > \rho_2 > \rho_3$ as required for stability. The bottom is a fixed horizontal plane, and the upper boundary at height $h_1+h_2+h_3$ above the bottom is also a fixed horizontal plane. The flow is two-dimensional in a vertical plane, and in each layer the primary velocity is c. We suppose that in the transition to a conjugate flow the lower interface is displaced from the height h_1 to the height $h_1+\phi_1$ above the bottom, and the upper interface is displaced from h_1+h_2 to $h_1+h_2-\phi_2$ (note the minus sign, which simplifies the argument later).

The conditions of energy conservation (Bernoulli's equation) for each layer give three equations relating ϕ_1, ϕ_2, the velocities in each layer and the pressure change at the top or bottom. The unknown velocities may be eliminated by means of the three mass-conservation conditions, and the unknown pressure change may then be eliminated among the three resulting equations. In this way one obtains

$$\phi_1 = \frac{1}{2\mu(\rho_1-\rho_2)}\left[\rho_1\left\{1-\left(\frac{h_1}{h_1+\phi_1}\right)^2\right\}-\rho_2\left\{1-\left(\frac{h_2}{h_2-\phi_1-\phi_2}\right)^2\right\}\right], \tag{5.1}$$

$$\phi_2 = \frac{1}{2\mu(\rho_2-\rho_3)}\left[-\rho_2\left\{1-\left(\frac{h_2}{h_2-\phi_1-\phi_2}\right)^2\right\}+\rho_3\left\{1-\left(\frac{h_3}{h_3+\phi_2}\right)^2\right\}\right], \tag{5.2}$$

where $\mu = g/c^2$. This pair of simultaneous equations may be expressed

$$\phi = \mu^{-1}\hat{A}\phi, \quad \text{with} \quad \phi = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}, \tag{5.3}$$

and the nonlinear transformation $\hat{A}$ may be considered as a mapping of the plane (u_1, u_2) into itself.

We focus attention on the possibility of a conjugate flow in which the lower interface is displaced upwards from its level in the primary flow and the upper interface is displaced downwards. In other words, we suppose that $\phi \in K$, where K is the first quadrant, $u_1 \geqslant 0$ and $u_2 \geqslant 0$. It is readily seen from (5.1) and (5.2) that $\hat{A}$ is a positive operator on K: that is,

$$\hat{A}u \in K \quad \text{if} \quad u \in K.$$

Thus it appears that the propositions explained earlier concerning nonlinear operators acting in cones may be applicable. We note also that $\hat{A}$ is monotonic, thus $\hat{A}u \leqslant \hat{A}v$ if $u \leqslant v$.

The derivative of $\hat{A}$ at the zero point θ may be understood from the definition of abstract differentiation noted in § 2. We see that $\hat{A}'(\theta)$ is a linear transformation of vectors ξ, which for the present purpose are taken in particular to belong to K. From (5.1) and (5.2) it is found that

$$\hat{A}'(\theta)\,\xi = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}, \tag{5.4}$$

in which
$$a_{11} = \frac{1}{\rho_1-\rho_2}\left(\frac{\rho_1}{h_1}+\frac{\rho_2}{h_2}\right), \quad a_{12} = \frac{1}{\rho_1-\rho_2}\left(\frac{\rho_2}{h_2}\right),$$
$$a_{21} = \frac{1}{\rho_2-\rho_3}\left(\frac{\rho_2}{h_2}\right), \quad a_{22} = \frac{1}{\rho_2-\rho_3}\left(\frac{\rho_2}{h_2}+\frac{\rho_3}{h_3}\right).$$

The eigenvalues of $\hat{A}'(\theta)$, i.e. the values of κ for which the equation

$$\kappa\xi = \hat{A}'(\theta)\,\xi \tag{5.5}$$

has a non-trivial solution, are the roots of

$$\det[\hat{A}'(\theta) - \kappa \boldsymbol{I}] = 0. \tag{5.6}$$

Hence the root corresponding to $\xi \in K$ is found to be

$$\tfrac{1}{2}(a_{11}+a_{22}) + \sqrt{\{\tfrac{1}{4}(a_{11}-a_{22})^2 + a_{12}a_{21}\}} = \mu_c, \quad \text{say}, \tag{5.7}$$

which obviously is always positive.

According to our general criterion, the primary flow is supercritical or subcritical in the considered mode accordingly as $\mu < \mu_c$ or $\mu > \mu_c$. And again, as in example 1, it can be seen that

$$C = \sqrt{(g/\mu_c)} \tag{5.8}$$

is the velocity, relative to the fluid, of infinitesimal long waves in this mode. Thus the condition of supercritical flow is $c > C$, as expected. For the other possible mode, in which the two interfaces are displaced in the same direction, the long-wave velocity is given by the second root of (5.6) and is larger than the present C.

Supposing that the primary flow is subcritical ($\mu > \mu_c$), we now show that a conjugate flow of the considered kind always exists and is supercritical. It is, of course, a fairly simple matter to tackle the algebraic equations (5.1) and (5.2) directly, but there is already in this example considerable interest in using the abstract theory. For the norm of planar vectors u we may take

$$\|u\| = |u_1| + |u_2|;$$

and for elements with small norms we have, confirming the definition of the derivative of $\hat{A}$,

$$\hat{A}u = \hat{A}'(\theta)\,u + \omega(u),$$

where
$$\lim_{\|u\|\to 0} \frac{|\omega(u)|}{\|u\|} = 0.$$

The assumption $\mu > \mu_c$ implies that

$$\mu^{-1}\hat{A}u - u \notin K \quad \text{if} \quad u \in K,\ \|u\| = r, \tag{5.9}$$

where r is a sufficiently small positive number; and thus the first condition (2.10) of theorem II (§ 2.2) is provided. For the proof of (5.9) we consider that, for any non-zero element of K, a finite positive number β_m may be defined as the *least* value of β for which $u \leqslant \beta\xi$. It is evident that

$$\|u\| \geqslant (1+b)^{-1}\,\|\beta_m \xi\|,$$

where
$$b = \max\,(\xi_1/\xi_2,\ \xi_2/\xi_1) < \infty.$$

Now suppose that (5.9) is not true. We then have, by virtue of the monotonicity of $\hat{A}$,

$$\begin{aligned}\mu u \leqslant \hat{A}u \leqslant \hat{A}(\beta_m\xi) &= \beta_m \hat{A}'(\theta)\,\xi + \omega(\beta_m\xi)\\ &= \beta_m \mu_c \xi + \omega(\beta_m \xi).\end{aligned} \tag{5.10}$$

The remainder ω can be made arbitrarily small by taking $\|u\|$ small enough. Thus, for small $\|u\|$ and $\mu > \mu_c$, the conclusion (5.10) contradicts the minimal property of β_m, and so the truth of (5.9) is established.

We next observe from (5.1) and (5.2) that Au is unbounded in the limit as $u_1 + u_2$ approaches the value h_2. A sufficiently small positive number δ can certainly be found, therefore, for which

$$u - \mu^{-1}\hat{A}u \notin K \quad \text{if} \quad u \in K,\ \|u\| = h_2 - \delta. \tag{5.11}$$

Furthermore, $\hat{A}$ is continuous in the region of K where $\|u\| \leqslant h_2 - \delta$. According to theorem II, this condition together with (5.9) and (5.11) ensure the existence of a non-trivial solution ϕ of (5.3) in K. Since $\hat{A}$ is neither concave nor convex everywhere, we cannot make a simple conclusion about uniqueness. However, the general argument of § 3.4 concerning the indices of fixed points in a cone guarantees the existence of a solution representing a supercritical conjugate flow.

An expression for the flow-force difference as a function of ϕ_1 and ϕ_2 can easily be obtained by the use of Bernoulli's equation to eliminate the pressure change at the top or bottom. Hence a straightforward study confirms the principle that a minimum value of flow force is realized by a supercritical conjugate flow.

6. Example 3. Continuously stratified fluids

We consider the parallel flow of a heterogeneous incompressible fluid between fixed horizontal boundaries $y = 0$ and $y = 1$ (figure 6). It is assumed that in the primary state the density ρ of the fluid is a non-increasing continuous function of height y, so that the system is statically stable. In cases where the primary velocity U is non-uniform additional assumptions are needed to ensure stability, but this aspect can be passed over here: a sufficient condition for stability is that $-g\,\mathrm{d}(\ln\rho)/\mathrm{d}y \geqslant \frac{1}{4}(\mathrm{d}U/\mathrm{d}y)^2$ (see Miles 1963). An approximate account of conjugate flows in continuously stratified fluids has already been given by Benjamin (1966, § 3.8), in the context of a theory of internal solitary and cnoidal waves. A more or less complete analogy with the theory of steady two-dimensional open-channel flows was demonstrated, including the principle that the range of a parameter (such as the value of flow force) closed by an adjacent pair of conjugate flows corresponds to a spectrum of periodic waves. The previous discussion was restricted, however, to states of flow close to critical.

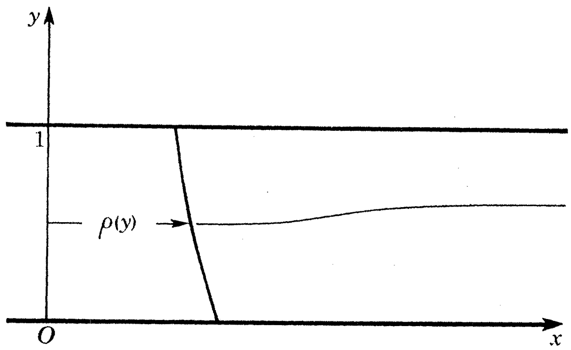

FIGURE 6. Illustration of continuously stratified fluid.

6.1. *Governing differential equation*

The most convenient choice of dependent variable is the pseudo-stream-function $\psi(y)$ defined by

$$\psi_y = \rho^{\frac{1}{2}}u, \quad \psi(0) = 0, \tag{6.1}$$

where u is the (horizontal) flow velocity. In a transition between conjugate flows density is conserved along each streamline, and this condition may be expressed as

$$\rho = \rho(\psi). \tag{6.2}$$

The dynamical condition satisfied by conjugate flows is that total head (stagnation pressure) is conserved along each streamline, thus

$$H = p + gy\rho + \tfrac{1}{2}\psi_y^2 = H(\psi). \tag{6.3}$$

Also, since the flows are horizontal, the pressure p satisfies the hydrostatic law

$$p_y = -g\rho. \tag{6.4}$$

Differentiating (6.3) and using (6.2) and (6.4), we obtain

$$\psi_{yy} + gy\rho'(\psi) - H'(\psi) = 0 \tag{6.5}$$

[cf. appendix 2, equation (A 36)].

The specifications of the primary flow, for which ψ is a known function $\Psi(y)$, say, can be used to determine the forms of the functions $\rho(\psi)$ and $H(\psi)$ appearing in (6.5), so that the equation reduces to an identity when $\psi = \Psi$. Hence a conjugate flow is represented by a solution of (6.5) differing from Ψ and satisfying the kinematical conditions

$$\psi(0) = 0, \quad \psi(1) = \Psi(1). \tag{6.6}$$

The problem becomes more clearly defined on the substitution of $\psi = \Psi + j\phi$, where j stands for a positive normalizing factor which can be chosen appropriately in particular examples. Then (6.5) is reducible to the form

$$\phi_{yy} + \phi f(y, \phi; \mu) = 0, \tag{6.7}$$

where μ is a positive parameter depending inversely on the velocity scale of the primary flow, and (6.6) becomes

$$\phi(0) = 0, \quad \phi(1) = 0. \tag{6.8}$$

The nonlinear boundary-value problem expressed by (6.7) and (6.8) has the trivial solution $\phi \equiv 0$, which represents the primary flow, and any other solution is significant as representing a flow conjugate to this.

Although a governing equation in the form (6.7) is an obvious outcome, the detailed reduction of (6.5) is a somewhat intricate matter. It seems worth while to go into the details, and to clarify the method of reduction by treating a specific example.

Let Y denote the height of particular streamlines in the primary flow. For this flow (6.4) gives

$$H'(\Psi) = \mathrm{d}\{p + gY\rho(\Psi) + \tfrac{1}{2}\Psi_y^2\}/\mathrm{d}\Psi = gY\rho'(\Psi) + \Psi_{YY}, \tag{6.9}$$

and thus we see that in order to reduce (6.5) in any specific case, Y and Ψ_{YY} as well as ρ must be expressed as functions of Ψ. The immediate result of substituting $\psi = \Psi + j\phi$ in (6.5) is

$$j\phi_{yy} = -gy\rho'(\Psi + j\phi) + H'(\Psi + j\phi) - \Psi_{yy};$$

and hence the use of (6.9) leads to

$$j\phi_{yy} = g\rho'(\Psi + j\phi)\{Y(\Psi + j\phi) - y\} + \Psi_{YY}(\Psi + j\phi) - \Psi_{yy}. \tag{6.10}$$

Noting the identities $y = Y(\Psi)$, $\Psi_{YY}(\Psi) = \Psi_{yy}$ and assuming that the functions $\rho'(\psi)$, $Y(\psi)$ and $\Psi_{YY}(\psi)$ are differentiable, or at least Lipschitz continuous, we conclude that (6.10) is equivalent to (6.7), in which $f(y, \phi; \mu)$ is a bounded function of ϕ. By checking dimensions we also see from (6.10) that $f(y, \phi; \mu)$ can generally be arranged in the form $\mu f_1(y, \phi) + f_2(y, \phi)$ with $\mu = g\beta/c^2$, where c is the velocity scale of the primary flow and β is a constant.

[Note that for certain special choices of primary flow equation (6.7) is *linear*; but such cases have no interest in present respects. For example, suppose that $\rho^{\frac{1}{2}}U = a$ (const.) in the primary flow. Then $\Psi = aY$, and so $\Psi_{YY} = 0$. Also,

$$Y(\Psi + j\phi) - y = -a^{-1}j\phi \quad \text{and} \quad \rho'(\Psi + j\phi) = \rho'(ay) = a^{-1}\rho_y(y).$$

Thus (6.10) reduces to $\phi_{yy} - (ga^{-2}\rho_y)\phi = 0$. Conjugate-flow pairs do not exist in linear systems, and such cases are also physically exceptional in other ways (see Benjamin 1966, § 3.9).]

To exemplify the usual case in which the governing equation (6.7) is nonlinear, we take the specification of the primary flow to be

$$U = c, \quad \rho = \rho_0\,\mathrm{e}^{-\beta Y}, \tag{6.11}$$

where c and ρ_0 are constants. It follows that

$$\Psi = 2\rho_0^{\frac{1}{2}}c\beta^{-1}(1 - \mathrm{e}^{-\frac{1}{2}\beta Y});$$

hence
$$\Psi_{YY} = -\tfrac{1}{2}\rho_0^{\frac{1}{2}}c\beta\,\mathrm{e}^{-\frac{1}{2}\beta Y} = -\tfrac{1}{2}\rho_0^{\frac{1}{2}}c\beta + \tfrac{1}{4}\beta^2\Psi, \tag{6.12}$$

and
$$Y(\Psi) = -\frac{2}{\beta}\ln\left(1 - \frac{\beta\Psi}{2\rho_0^{\frac{1}{2}}c}\right). \tag{6.13}$$

Using (6.12) and (6.13) in (6.10), and taking $j = 2\rho_0^{\frac{1}{2}}c/\beta$, we obtain

$$\phi_{yy} + \phi\{\mu Z(\mathrm{e}^{\frac{1}{2}\beta y}\phi) - \tfrac{1}{4}\beta^2\} = 0, \tag{6.14}$$

where $\mu = g\beta/c^2$ and the function with argument $\mathrm{e}^{\frac{1}{2}\beta y}\phi$ is

$$Z(z) = -\frac{(1-z)\ln(1-z)}{z}. \tag{6.15}$$

We note that $Z(z)$ is a positive, monotonic decreasing function for $z < 1$, and that $Z(0) = 1$.

6.2. *Extension of properties outside range of primary flow*

To cover the case of conjugate flows greatly different from the primary flow, allowance must be made for the possibility that $\psi(y)$ takes values outside the interval $[0, \Psi(1)]$ over which the primary-flow properties are defined. This means that the conjugate flow partially consists of an 'eddy' of fluid that is not present in the primary flow. An analytical continuation of $\rho(\psi)$ and $H(\psi)$ provides a poor physical model, because fluid at the centre of the eddy is then taken to be lighter than fluid above and this is an unstable situation; so other ways of extending the flow properties are preferable. Perhaps the best model is given by specifying that new fluid appearing in a conjugate flow has constant density: thus $\rho = \rho(0)$ if $\psi \leqslant 0$ and $\rho = \rho\{\Psi(1)\}$ if $\psi \geqslant \Psi(1)$. This specification is physically reasonable as it reflects the mixing process that is likely to occur within a closed eddy in practice; and it can be made consistently with the assumption that $f(y, \phi; \mu)$ in (6.7) is a continuous function of ϕ. In the example given above, the condition $\psi(y) = \Psi(1)$ for $0 < y < 1$ is equivalent to

$$\phi(y) = \mathrm{e}^{-\frac{1}{2}\beta y} - \mathrm{e}^{-\frac{1}{2}\beta} < 1.$$

Hence, on the assumption that $\mu > \frac{1}{4}\beta^2$, an extension of the function $f = \mu Z(\mathrm{e}^{\frac{1}{2}\beta y}\phi) - \frac{1}{4}\beta^2$ for $\phi \geqslant \mathrm{e}^{-\frac{1}{2}\beta y} - \mathrm{e}^{-\frac{1}{2}\beta}$ can be made such that the modified function is continuous, positive and monotonic for all ϕ. The details are immaterial to the existence proof given below, for which we only need to assume that

$$f(y, \phi; \mu) \leqslant \sigma_m \quad \text{if} \quad \phi \geqslant q_m(y), \tag{6.16}$$

where σ_m is a finite positive constant and $q_m(y)$ is a bounded positive function on $[0, 1)$, representing some arbitrary degree of penetration into the eddy region. A similar modification can be made for $\psi < 0$, which in the given example means

$$\phi < -(1 - \mathrm{e}^{-\frac{1}{2}\beta y}).$$

We assume that

$$f(y, \phi; \mu) \geqslant \sigma_M \quad \text{if} \quad \phi \leqslant -q_M(y), \tag{6.17}$$

where $q_M(y)$ is bounded and positive on $(0, 1]$.

It may be expected, however, that when μ is close to a critical value the conjugate solution ϕ will be small everywhere on $(0, 1)$, and therefore these extensions of equation (6.7) will be unnecessary. A method for testing the corresponding property of conjugate vortex flows is given in Benjamin (1971, § 6), and is easily adaptable to the present problem.

6.3. *Governing equation in operator form*

In order to express the problem in a form to which the abstract theory is applicable, the system (6.7), (6.8) is recast as the nonlinear integral equation of Hammerstein type

$$\phi(y) = \int_0^1 k(y, \hat{y})\, \phi(\hat{y}) f\{\hat{y}, \phi(\hat{y}); \mu\}\, \mathrm{d}\hat{y}, \tag{6.18}$$

in which $k(y, \hat{y})$ is the triangular Green function

$$k(y, \hat{y}) = y(1-\hat{y}) \quad \text{if} \quad 0 \leqslant y \leqslant \hat{y},$$
$$= \hat{y}(1-y) \quad \text{if} \quad \hat{y} \leqslant y \leqslant 1.$$

This equation is written for short

$$\phi = \boldsymbol{A}(\mu)\,\phi; \tag{6.18'}$$

and the operator $\boldsymbol{A}$, written $\boldsymbol{A}(\mu)$ when a reminder of its dependence on the parameter μ is needed, may be expressed in the form

$$\boldsymbol{A} = \boldsymbol{BF}, \tag{6.19}$$

where $\boldsymbol{B}$ stands for the self-adjoint linear operator defined by

$$\boldsymbol{B}u(y) = \int_0^1 k(y, \hat{y})\, u(\hat{y})\, \mathrm{d}\hat{y}, \tag{6.20}$$

and $\boldsymbol{F}$ denotes the nonlinear operator defined by

$$\boldsymbol{F}u(y) = u(y) f\{y, u(y); \mu\}. \tag{6.21}$$

The specification that $f(y, u; \mu)$ is a continuous function of y and u ensures that $\boldsymbol{A}$ is a completely continuous operator in the space of continuous real-valued functions $C(0, 1)$ (K (*a*), p. 350; K (*b*), pp. 19 and 46). A solution of (6.18) is then twice continuously differentiable and so is simultaneously a solution of (6.7) and (6.8).

Furthermore, $\boldsymbol{A}$ has a strong Fréchet derivative in all directions of C, which is a completely continuous linear operator in C (K (*b*), p. 135). In particular, the derivative of $\boldsymbol{A}$ at the zero point θ of C, in the arbitrary direction h, is seen to be

$$\boldsymbol{A}'(\theta; \mu)\, h(y) = \boldsymbol{B}[h(y) f(y, 0; \mu)]. \tag{6.22}$$

Characteristic values $\mu_c^{(n)}$ $(n = 0, 1, 2, \ldots)$ may be defined as those values of μ for which the linear equation

$$h_n = \boldsymbol{A}'(\theta; \mu_c^{(n)})\, h_n \tag{6.23}$$

has a non-trivial solution $h_n \in C$. (It can easily be confirmed that these characteristic values are simple.) This equation is the linearized version of (6.18), and its solutions represent infinitesimal waves of extreme length than can be superposed on the primary flow when its velocity scale takes particular values. For example, in the case of an exponential density profile considered above, (6.23) is just

$$h_n = (\mu_c^{(n)} - \tfrac{1}{4}\beta^2)\, \boldsymbol{B}h_n,$$

which gives

$$\mu_c^{(n)} = (n+1)^2 \pi^2 + \tfrac{1}{4}\beta^2, \quad h_n = \sin\,(n+1)\,\pi y. \tag{6.24}$$

In view of the way the parameter μ enters the function $f(y, 0; \mu)$ [see remark below (6.10)], it is seen that in general, as in the given example, (6.23) is the inverse of a Sturm–Liouville system, and so the functions h_n are characterized by having exactly n zeros in the open interval (0, 1). Possible solutions ϕ of the nonlinear equation (6.18) may also be ordered in this sense.

We focus attention on the first mode $n = 0$, in which like h_0 a solution of (6.18) has no zero in the open interval. Our general criterion (§ 3.1) now shows that, with respect to this mode, the primary flow is

$$\left.\begin{array}{ll} \textit{supercritical} & \text{if} \quad \mu < \mu_c, \\ \textit{subcritical} & \text{if} \quad \mu > \mu_c. \end{array}\right\} \tag{6.25}$$

In the given example these conditions are the same as $c > C_0$ and $c < C_0$, respectively, where $C = \surd\{g\beta/(\pi^2 + \tfrac{1}{4}\beta^2)\}$ is the velocity of infinitesimal long waves in the first mode [cf. Benjamin 1966, equation (4.16)].

6.4. *Existence of conjugate flows*

We assume that, as in the given example, the continuous function $f(y,u;\mu)$ is positive and $\partial f/\partial\mu > 0$ for $0 \leqslant y \leqslant 1$, $\mu > 0$ and all u. The Green function $k(y,\hat{y})$ also is positive on the open square $0 < y < 1$, $0 < \hat{y} < 1$ and vanishes for $y = 0$ and $y = 1$. Hence we see that $\boldsymbol{A}$ transforms non-negative continuous functions $u(y)$ into non-negative, twice differentiable functions $\boldsymbol{A}u(y)$ which vanish at both end-points of $[0, 1]$ and are negatively curved on the open interval. Thus these functions belong to the cone $K \subset C$ of functions $v(y)$ that are convex upward on $[0, 1]$ and satisfy $v(0) = v(1) = 0$ (cf. K (*c*), § 7.4.6). By convex upward we mean that for any pair of points y_1, y_2 from $[0, 1]$ and any number α such that $0 < \alpha < 1$, we have

$$v\{\alpha y_1 + (1-\alpha)\, y_2\} \geqslant \alpha v(y_1) + (1-\alpha)\, v(y_2).$$

Convex functions are necessarily continuous, but not necessarily continuously differentiable: thus the closure of K includes functions with upward pointing corners. Evidently K is transformed into itself by the completely continuous operator $\boldsymbol{A}$, and we note that h_0 defined by (6.23) is an element of K. Although this cone is not solid in the space C, the arguments given at the end of § 3.4 establish the applicability of the conclusions that were drawn from index theory earlier in § 3.4 [e.g. the indices γ_j appearing in (3.24) or (3.25) can be evaluated by the standard formula (2.15)].

Suppose first that the primary flow is *subcritical* ($\mu > \mu_c$). As considered in § 3.1, let λ_θ denote the eigenvalue of $\boldsymbol{A}'(\theta;\mu)$ to which an eigenvector $\xi \in K$ corresponds, thus

$$\lambda_\theta \xi = \boldsymbol{A}'(\theta;\mu)\,\xi.$$

The properties (i) to (iv) of linear positive operators that were explained at the end of § 2.1 and discussed further in § 3.1 may be confirmed for the present $\boldsymbol{A}'(\theta;\mu)$: in particular, (i) the positive eigenvector ξ exists and (iii) is unique. We also note that $\boldsymbol{A}'(\theta;\mu)$ is self-adjoint with respect to the positive weight function $f_0 = f(y, 0;\mu)$ [see context of (2.21)]. Taking account of the assumed form of dependence on μ, and using the property of self-adjointness, we deduce

$$0 < \langle \xi, h_0\rangle_{f_0} = \langle \xi, \boldsymbol{A}'(\theta;\mu_c)\, h_0\rangle_{f_0} < \langle \xi, \boldsymbol{A}'(\theta;\mu)\, h_0\rangle_{f_0}$$
$$= \langle h_0, \boldsymbol{A}'(\theta;\mu)\,\xi\rangle_{f_0} = \lambda_\theta\langle h_0, \xi\rangle_{f_0},$$

which shows that

$$\lambda_\theta > 1. \tag{6.26}$$

Thus the condition of theorem III (§ 2.2) regarding the eigenvalue is satisfied. And we know, by the property (iii) already noted, that $\boldsymbol{A}'(\theta;\mu)$ has no positive eigenvector corresponding to an eigenvalue of unity.

The remaining condition of the fixed-point theorem may be established by use of the assumption (6.16). The upward convexity of any element v of K implies that $v \to \infty$ on $(0, 1)$ if $\|v\| \to \infty$, where $\|v\|$ is the norm for C:

$$\|v\| = \max_{y\in[0,1]} |v(y)|.$$

Hence (6.16) implies that in the limit $f\{y, v(y);\mu\}$ nowhere exceeds σ_m on $(0, 1)$. It follows from the continuity of f that $\boldsymbol{A}$ has a strong asymptotic derivative with respect to the cone K, and this is a linear positive operator $\boldsymbol{A}'(\infty)$ having the property that

$$\boldsymbol{A}'(\infty)\, w \leqslant \sigma_m w \quad \text{if} \quad w \in K. \tag{6.28}$$

The positiveness of $f(y, v;\mu)$ for all $v \geqslant \theta$ further implies that $\boldsymbol{A}'(\infty)$ has an eigenvector in K (K (*c*), § 2.2). Let λ_∞ denote the eigenvalue of $\boldsymbol{A}'(\infty)$ to which this positive eigenvector corresponds. Since $1/\pi^2$ is the eigenvalue of $\boldsymbol{B}$ to which the positive eigenvector $\sin \pi y$ corresponds, we readily find from (6.28), using the self-adjointness of $\boldsymbol{B}$, that

$$\lambda_\infty < 1 \quad \text{if} \quad \sigma_m < \pi^2. \tag{6.29}$$

Thus the condition of theorem III regarding the eigenvalue λ_∞ is satisfied if $\sigma_m < \pi^2$.

The theorem establishes, therefore, that if the primary flow is subcritical and if $\sigma_m < \pi^2$ in the specification (6.16), then equation (6.18) has at least one non-zero solution in K. Thus a conjugate flow exists whose streamlines are displaced downwards from their levels in the primary flow [i.e. $\psi = \Psi + j\phi > \Psi$ on $(0, 1)$]. Uniqueness can be proved only with an additional assumption, and this aspect will be discussed presently. If $\boldsymbol{F}$ and hence $\boldsymbol{A}$ is monotonic, however, the general arguments of § 3.4 show that among a multiplicity of fixed points in K there is at least one for which $\lambda_\phi < 1$, where λ_ϕ is the eigenvalue of $\boldsymbol{A}'(\phi; \mu)$ to which a positive eigenvector corresponds. The fact that $\boldsymbol{A}$ is monotonic also implies that $\boldsymbol{A}'(\phi; \mu)$ is a positive linear operator, being self-adjoint with respect to a positive weight function. Hence an obvious adaptation of the argument used above to prove $\lambda_\theta > 1$ shows that the fixed point for which $\lambda_\phi < 1$ represents a supercritical conjugate flow (i.e. μ would have to be increased to make this flow critical).

If the function $f(y, u; \mu)$ is *monotonic decreasing* in u, as in the given example, it can be shown that the non-zero solution ϕ is *unique*. In this case $\boldsymbol{F}$ and hence $\boldsymbol{A}$ is a concave operator, and so the demonstration of uniqueness given in § 3.4 may be applicable. In § 3.4, however, it was supposed additionally that $\boldsymbol{A}'(\phi; \mu)$ is self-adjoint with respect to a positive weight function, which here requires that $f \geqslant -\phi f_\phi$ [see (6.30) and (6.31) below]. A more general conclusion is obtained as follows. An expression defining the linear operator $\boldsymbol{A}'(\phi; \mu)$ is found to be

$$\boldsymbol{A}'(\phi; \mu)\,\eta = \boldsymbol{B}[(f + \phi f_\phi)\,\eta], \tag{6.30}$$

where f stands for $f\{y, \phi(y); \mu\}$ and

$$\phi f_\phi \equiv \left[z\,\frac{\partial f(y, z; \mu)}{\partial z}\right]_{z=\phi(y)}. \tag{6.31}$$

Although $\boldsymbol{A}'(\phi; \mu)$ is not necessarily a positive operator on K unless $f \geqslant -\phi f_\phi$, we may assume that it nevertheless has a positive eigenvector η corresponding to an eigenvalue λ_ϕ which is higher than any other eigenvalue. Considering that $\phi = \boldsymbol{A}\phi = \boldsymbol{B}(f\phi)$ and using the self-adjointness of $\boldsymbol{B}$, we deduce

$$\begin{aligned}(1-\lambda_\phi)\,\langle f\eta, \phi\rangle &= \langle f\eta, \boldsymbol{B}(f\phi)\rangle - \langle f\phi, \boldsymbol{B}[(f+\phi f_\phi)\,\eta]\rangle \\ &= -\langle f\phi, \boldsymbol{B}(\phi f_\phi \eta)\rangle \equiv -\int_0^1 f\phi\,\boldsymbol{B}(\phi f_\phi \eta)\,\mathrm{d}y = s, \quad \text{say.}\end{aligned} \tag{3.32}$$

In the present case we have that ϕf_ϕ is non-positive, and we may assume that η is positive on $(0, 1)$ and that ϕf_ϕ is not zero everywhere on $(0, 1)$. Hence $s > 0$ and so $\lambda_\phi < 1$. Recalling § 3.4, we conclude that $\gamma = 1$, where γ is the index of the non-zero fixed point ϕ, and therefore, according to theorem A (§ 3.4), the fixed point is unique. Thus only a single conjugate flow is possible in the present special case. (The impossibility of a non-positive solution is evident from the simple considerations pointed out in § 3.2.)

In the case of a *supercritical* primary flow ($\mu < \mu_c$), the existence of a conjugate flow may be established by means of theorem IV (§ 2.2). We put $\phi = -\bar{\phi}$ and consider the possibility of a non-positive solution ϕ by assuming $\bar{\phi} \in K$. The equation for $\bar{\phi}$ is $\bar{\phi} = \bar{\boldsymbol{A}}\bar{\phi}$, where $\bar{\boldsymbol{A}}$ is the positive operator defined by $\bar{\boldsymbol{A}}u = -\boldsymbol{A}(-u)$ for $u \in K$. An argument corresponding to that leading to

(6.26) shows now that $\lambda_0 < 1$, and thus one condition of theorem IV is satisfied. Next it may be shown that the operator $\bar{\boldsymbol{A}}$ has a strong asymptotic derivative with respect to the cone K, and by virtue of the condition (6.17) this has the property

$$\bar{\boldsymbol{A}}'(\infty)\, w \geqslant \sigma_M \boldsymbol{B} w \quad \text{if} \quad w \in K. \tag{6.33}$$

Assuming that

$$\sigma_M > \pi^2, \tag{6.34}$$

and recalling that f is a bounded function (see § 6.2), we may conclude that $\bar{\boldsymbol{A}}'(\infty)$ has an eigenvector in K corresponding to an eigenvalue $\lambda_\infty > 1$. Also, $\bar{\boldsymbol{A}}'(\infty)$ does not have an eigenvector in K corresponding to an eigenvalue of unity. Thus all the conditions of theorem IV are satisfied and the existence of a conjugate flow follows.

As was explained in § 3.4 concerning the interpretation of theorem B, the deduction just made can be supplemented by the statement that at least one non-zero fixed point $\bar{\phi} \in K$ exists for which $\bar{\boldsymbol{A}}'(\bar{\phi};\mu)$ has an eigenvalue $\lambda_\phi > 1$. Hence we may infer that a subcritical conjugate flow exists. If, as in the given example [see (6.14) and (6.15)], $f(y, -u;\mu)$ is a *monotonic increasing* function of u, then $\bar{\boldsymbol{A}}$ is a convex operator on K. Consequently, as explained in § 3.4, the fixed point $\bar{\phi}$ is unique if no eigenvalue of $\bar{\boldsymbol{A}}'(\bar{\phi};\mu)$ other than λ_ϕ can exceed unity. Without going into details, we note that this condition is provided if

$$f + u f_u < 4\pi^2 \quad \text{for} \quad u \leqslant 0. \tag{6.35}$$

The significance of the number $4\pi^2$ appearing here is that $(4\pi^2)^{-1}$ is the second eigenvalue of $\boldsymbol{B}$ defined by (6.20) (cf. Benjamin 1971, § 5).

6.5. *Flow force*

Here we illustrate the general theory of flow-force differences between conjugate flows that was developed in § 3.5. In particular, a detailed application is made of one of the variational proofs outlined in § 3.5, namely that establishing the existence of a supercritical conjugate flow corresponding to a subcritical primary flow. We now consider the solution ϕ of (6.18) as an element of the Hilbert space $L_2(0, 1)$. The cone K previously defined in C is evidently a cone in this space also. By virtue of the fact that the continuous function f is bounded, $\boldsymbol{F}$ defined by (6.21) is a continuous nonlinear operator acting in L_2; and the linear operator $\boldsymbol{B}$ defined by (6.20) is completely continuous acting from L_2 into $C \subset L_2$ (K (b), p. 19). It follows that the Hammerstein operator $\boldsymbol{A}$ is completely continuous in the same respect. Although $\boldsymbol{A}$ is not a potential operator, the problem can be put into variational form by means of the well-known device mentioned in § 3.5 (K (b), p. 304; Vainberg 1964, ch. 7).

The self-adjoint linear operator $\boldsymbol{B}$ is positive in the special sense that

$$\langle u, \boldsymbol{B}u \rangle > 0 \quad \text{for} \quad u \in L_2,\ u \neq \theta, \tag{6.36}$$

which corresponds to the fact that the eigenvalues $(m\pi)^2$ $(m = 1, 2, \ldots)$ of $\boldsymbol{B}$ are all positive. It follows that a self-adjoint operator $\boldsymbol{B}^{\frac{1}{2}}$, positive in the above sense, can be defined such that $\boldsymbol{B}^{\frac{1}{2}}(\boldsymbol{B}^{\frac{1}{2}}u) = \boldsymbol{B}u$ [and so $\langle u, \boldsymbol{B}u \rangle = \langle \boldsymbol{B}^{\frac{1}{2}}u, \boldsymbol{B}^{\frac{1}{2}}u \rangle$]. In fact, since the set of eigenvectors of $\boldsymbol{B}$ is just the complete orthogonal set of sine functions vanishing at the end-points of $[0, 1]$, we see that

$$k(y, \hat{y}) = 2 \sum_{m=1}^{\infty} \frac{\sin m\pi y \, \sin m\pi \hat{y}}{m^2\pi^2}, \tag{6.37}$$

and hence that the explicit form of $\boldsymbol{B}^{\frac{1}{2}}$ is given by replacing k in (6.20) with

$$k_{\frac{1}{2}}(y, \hat{y}) = 2 \sum_{m=1}^{\infty} \frac{\sin m\pi y \sin m\pi \hat{y}}{m\pi}$$

(cf. K (*b*), p. 51). This kernel has a logarithmic singularity on the diagonal but still satisfies a sufficient condition (K (*b*), p. 19) for the operator $\boldsymbol{B}^{\frac{1}{2}}$ to be completely continuous in L_2.

Since $\boldsymbol{B}^{\frac{1}{2}}u = \theta$ only if $u = \theta$, it follows that if we put $\phi = \boldsymbol{B}^{\frac{1}{2}}\zeta$ equation (6.18) is reducible to

$$\zeta = \boldsymbol{B}^{\frac{1}{2}}\boldsymbol{F}(\boldsymbol{B}^{\frac{1}{2}}\zeta). \tag{6.38}$$

That is, if ζ is established as a solution of (6.38), then $\boldsymbol{B}^{\frac{1}{2}}\zeta$ is a solution of (6.18). We consider a modification of (6.38) in the form

$$\zeta = \boldsymbol{B}^{\frac{1}{2}}\boldsymbol{F}(|\boldsymbol{B}^{\frac{1}{2}}\zeta|). \tag{6.39}$$

The operator on the right-hand side of (6.39) is completely continuous, and it differs from that on the right-hand side of (6.38) only if $\boldsymbol{B}^{\frac{1}{2}}\zeta$ takes negative values. But corresponding to any non-zero solution of (6.39) we have

$$\phi = \boldsymbol{B}^{\frac{1}{2}}\zeta = \boldsymbol{BF}(|\boldsymbol{B}^{\frac{1}{2}}\zeta|) \in K,$$

by virtue of the fact that $\boldsymbol{B}$ maps non-negative functions into the cone K previously defined. Thus, for the study of solutions of (6.18) that are non-zero elements of K, (6.39) suffices in place of (6.38).

We define

$$W(u) = \int_0^u \boldsymbol{F}(|z|)\,\mathrm{d}z = \int_0^u |z| f(y, |z|; \mu)\,\mathrm{d}z, \tag{6.40}$$

and observe that the operator on the right-hand side of (6.39) is the gradient of the functional

$$\Omega(u) = \int_0^1 W(\boldsymbol{B}^{\frac{1}{2}}u)\,\mathrm{d}y \tag{6.41}$$

(K (*b*), p. 71). Equation (6.39) is therefore equivalent to $\operatorname{grad} \Lambda(\zeta) = \theta$, where

$$\Lambda(u) = \tfrac{1}{2}\langle u, u\rangle - \Omega(u). \tag{6.42}$$

The argument outlined in §§ 2.5 and 3.5 may now be used. By virtue of the complete continuity of $\boldsymbol{B}^{\frac{1}{2}}$, the functional (6.41) is weakly continuous in L_2. Also, $\langle u, u\rangle = \|u\|^2_{L_2}$ is weakly lower semi-continuous and hence so is $\Lambda(u)$. It follows that on any bounded and weakly closed subset of L_2, Λ has an infimum which it achieves (Vainberg 1964, p. 78). And if the infimum is achieved at an interior point ζ, then $\Lambda(\zeta)$ is a minimum in the usual sense and therefore $\operatorname{grad} \Lambda(\zeta) = \theta$. Considering the closed ball $0 \leqslant \|u\| \leqslant R$ in L_2, we thus see that the existence of a non-trivial solution of (6.39) is proven if it can be shown, first, that $\Lambda(\theta) = 0$ is not a minimum and, secondly, that $\Lambda(u)$ is positive on the spherical boundary $\|u\| = R$.

It was explained in § 3.5 that $\Lambda(\theta)$ cannot be a minimum if the primary flow is *subcritical* ($\lambda_\theta > 1$). For then Λ can be shown to take some negative values at points infinitesimally close to θ, specifically points on the line $t\xi$, where ξ is the positive eigenvector of $\boldsymbol{A}'(\theta; \mu)$. We note that the modification of the operator entailed in (6.39) does not affect this conclusion. Thus the first condition of the existence theorem is provided by assuming the primary flow to be subcritical.

The assumption (6.16) leads easily to the inequality

$$W(u) \leqslant W(q_m) - \tfrac{1}{2}\sigma_m q_m^2 + \tfrac{1}{2}\sigma_m u^2 \tag{6.43}$$

if $f(y, u; \mu)$ is a monotonic decreasing function of u, as in the given example. Otherwise an upper bound for $W(u)$ can still be found in this form, namely as the sum of $\frac{1}{2}\sigma_m u^2$ and a fixed function. Hence we obtain from (6.42), using the self-adjointness of $\boldsymbol{B}^{\frac{1}{2}}$,

$$\Lambda(u) \geqslant \tfrac{1}{2}\langle u, u\rangle - \tfrac{1}{2}\sigma_m\langle u, \boldsymbol{B}u\rangle + a, \tag{6.44}$$

where a is a fixed number. The further inequality

$$\langle u, \boldsymbol{B}u\rangle \leqslant \pi^{-2}\langle u, u\rangle$$

appears as a property of the operator $\boldsymbol{B}$ when the kernel k is expressed in the form (6.37) and the L_2 (Plancherel) statement of Fourier's theorem is considered. Accordingly it follows from (6.44) that if $\sigma_m < \pi^2$ [cf. (6.29)], then $\Lambda(u) > 0$ on a sphere $\|u\| = R$ for which R is sufficiently large.

Thus the final condition of the variational existence theorem is satisfied if $\sigma_m < \pi^2$. We conclude that a non-zero element ζ of L_2 exists satisfying (6.39), and consequently also (6.38), so that $\phi = B^{\frac{1}{2}}\zeta \in K$ is a solution of the original problem. We know that

$$\Lambda(\zeta) = \min_{0\leqslant\|u\|_{L_2}\leqslant R} \Lambda(u) < 0; \tag{6.45}$$

and by the principle of minimum flow force (§ 3.5) the conjugate flow represented by ζ and ϕ must be supercritical. If the solution ϕ is not a unique fixed point of $\boldsymbol{A}$ in K, then the present result enables us to define the *principal* conjugate as the necessarily supercritical flow that realizes the absolute minimum of Λ.

It remains to show that Λ is in fact proportional to the flow-force difference between the conjugate and primary flow. By virtue of being a solution of (6.18), $\phi(y)$ is a twice differentiable function. Hence, putting $\zeta = \boldsymbol{B}^{-\frac{1}{2}}\phi$, where $\boldsymbol{B}^{-\frac{1}{2}}$ is the self-adjoint pseudo-differential operator such that $\boldsymbol{B}^{\frac{1}{2}}(\boldsymbol{B}^{-\frac{1}{2}}\phi) = \phi$, we obtain

$$\begin{aligned}\langle \zeta, \zeta\rangle &= \int_0^1 (\boldsymbol{B}^{-\frac{1}{2}}\phi)^2\,\mathrm{d}y = \int_0^1 \phi \boldsymbol{B}^{-1}\phi\,\mathrm{d}y\\ &= \int_0^1 \phi(-\phi_{yy})\,\mathrm{d}y = \int_0^1 \phi_y^2\,\mathrm{d}y.\end{aligned}$$

(For the last equality the boundary conditions (6.8) satisfied by ϕ are used in an integration by parts.) Thus an expression for $\Lambda(\zeta)$ is seen from (6.42) to be

$$\Lambda(\zeta) = \int_0^1 \{\tfrac{1}{2}\phi_y^2 - W(\phi)\}\,\mathrm{d}y = \Lambda_1(\phi), \quad \text{say;} \tag{6.46}$$

and we may express the differential equation (6.7) as

$$-\operatorname{grad} \Lambda_1(\phi) = \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial \phi_y}\right) - \frac{\partial I}{\partial \phi} = 0, \tag{6.47}$$

where $I(y, \phi, \phi_y)$ denotes the integrand in (6.46). That is, (6.47) is the Euler–Lagrange equation for the functional $\Lambda_1(\phi)$.

Now, the flow force S is defined as the sum of horizontal pressure force and momentum flux through a cross-section of the flow, thus

$$S = \int_0^1 (p + \psi_y^2)\,\mathrm{d}y. \tag{6.48}$$

Using (6.3) to eliminate the pressure p, we obtain from (6.48)

$$S = \int_0^1 \{H(\psi) + \tfrac{1}{2}\psi_y^2 - gy\rho(\psi)\}\,\mathrm{d}y, \tag{6.49}$$

which expresses S as a functional of the pseudo-stream-function $\psi(y)$. The differential equation (6.5) satisfied by ψ is

$$-\operatorname{grad} S(\psi) = 0; \tag{6.50}$$

and we recall that equation (6.7) was obtained by putting $\psi = \Psi + j\phi$ in (6.5) and subtracting the same equation for the primary solution Ψ. That is, we have

$$\operatorname{grad} S(\Psi + j\phi) - \operatorname{grad} S(\Psi) = j \operatorname{grad} \Lambda_1(\phi). \tag{6.51}$$

From (6.51) and the fact that $\Lambda_1(\theta) = 0$ there follows

$$j\Lambda_1(\phi) = S(\Psi + j\phi) - S(\Psi), \tag{6.52}$$

which is the required identity.

The result (6.45) is thus seen to imply that, corresponding to a subcritical primary flow, the principal conjugate flow manifests a minimum value of S less than the value for the primary flow.

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Appendix 1. Proofs of theorem A and theorem B

We recall from § 3.4 that in the statements of these theorems $\tilde{A}$ is a completely continuous operator which maps the whole of the Banach space E into a cone $K \subset E$, and which is identical with the completely continuous positive operator A on K [see (3.18) and (3.19)]. Also S_r and S_R stand for spheres $\|u\| = r > 0$ and $\|u\| = R > r$ centred on the point θ. The first theorem is restated as follows for easy reference.

Theorem A. *Let the conditions of theorem* I (§ 2.2) *be satisfied, or let the conditions of theorem* III (§ 2.2) *be satisfied and in addition let the cone K be normal. The there exist in E spheres S_r and S_R on which the rotations of the completely continuous vector field $I - \tilde{A}$ are respectively*

$$\gamma(S_r) = 0, \tag{A 1}$$

and

$$\gamma(S_R) = 1. \tag{A 2}$$

Suppose first that the conditions of theorem I are satisfied. We then have

$$u - \tilde{A}u \notin K \quad \text{on} \quad S_r. \tag{A 3}$$

This property is obvious for points on S_r such that $u \notin K$, since by definition $\tilde{A}u \in K$ for all $u \in E$. And on the intersection of S_r with K the property is provided by the condition (2.8) of theorem I. According to (A 3) none of the vectors $u - \tilde{A}u$ on S_r vanishes, and none has the same direction as any element of the cone. It follows that, as expressed by (A 1), the rotation of the completely continuous vector field $I - \tilde{A}$ on S_r is zero.

On the larger sphere S_R the field turns out to be homotopic to I, and therefore its rotation is equal to unity. To establish this it is sufficient to show that the completely continuous vector fields

$$F(u, t) = u - t\tilde{A}u$$

do not vanish anywhere on S_R for $0 < t \leqslant 1$. We assume to the contrary that there exists a vector $u_0 \in S_R$ such that $F(u_0, t) = \theta$ and so

$$u_0 = t\tilde{A}u_0 \in K.$$

By the definition of $\tilde{A}$ [see (3.19)] this implies that $\tilde{A}u_0 = Au_0$, and therefore

$$t(Au_0 - u_0) = (1 - t)\, u_0 \in K,$$

which contradicts the condition (2.9) of theorem I. Hence we conclude that (A 2) holds.

Suppose next that the conditions of theorem III are satisfied, and that K is a normal cone. We may establish (A 1) by showing that, if r is sufficiently small, there cannot exist a vector $u_1 \in S_r$ such that

$$u_1 - \tilde{A}u_1 = a\xi \quad (a \geqslant 0), \tag{A 4}$$

where ξ ($\|\xi\| = 1$) is the normalized positive eigenvector corresponding to the eigenvalue $\lambda_\theta > 1$ of the strong Fréchet derivative $A'(\theta)$ with respect to the cone. The impossibility of (A 4) on S_r means first (in the case $a = 0$) that the field $I - \tilde{A}$ does not vanish and therefore its rotation is defined, and secondly (in the case $a > 0$) that no component of the field has the same direction as the positive vector ξ. Hence the rotation of the field is zero.

We prove this by contradiction, assuming the existence of a vector $u_1 \in S_r$ which satisfies (A 4). Since $\tilde{A}u_1 \in K$ by the definition of $\tilde{A}$, it then follows from (A 4) that $u_1 \in K$ and so $\tilde{A}u_1 = Au_1$. Thus (A 4) amounts to

$$\left.\begin{aligned} u_1 - Au_1 &= a\xi \quad (a \geqslant 0), \\ \text{with} \qquad u_1 &\in K \quad (\|u_1\| = r). \end{aligned}\right\} \tag{A 5}$$

And on the substitution of

$$\psi = u_1 + \frac{a\xi}{\lambda_\theta - 1}, \tag{A 6}$$

which is a positive vector since $\lambda_\theta > 1$, the equation in (A 5) becomes

$$\psi - A'(\theta)\,\psi = Au_1 - A'(\theta)\,u_1. \tag{A 7}$$

Now, a condition of theorem III is that $A'(\theta)$ does not have a positive eigenvector to which an eigenvalue of unity corresponds, and this implies that a number $\alpha > 0$ can be found such that

$$\|\psi - A'(\theta)\,\psi\| \geqslant \alpha\|\psi\| \quad \text{if} \quad \psi \in K. \tag{A 8}$$

Also, according to the definition of a strong derivative with respect to a cone [see sentence following (2.3)], a number $\rho(p) > 0$ can be found such that $\|u_1\| = r \leqslant \rho(p)$ implies

$$\|Au_1 - A'(\theta)\,u_1\| \leqslant p\alpha\|u_1\| \quad \text{if} \quad u_1 \in K, \tag{A 9}$$

where p is a given positive number. Hence (A 7) shows that

$$\|\psi\| \leqslant p\|u_1\| \quad \text{if} \quad u_1 \in K, \quad 0 < r \leqslant \rho(p). \tag{A 10}$$

But (A 6) also implies that $\psi - u_1 \in K$, and therefore, since the cone K is normal (see remarks following the definition of a normal cone in § 2.1), we have

$$N\|\psi\| \geqslant \|u_1\| \quad \text{if} \quad u_1 \in K. \tag{A 11}$$

The inequalities (A 10) and (A 11) are contradictory if the arbitrary number p is given any value satisfying $p < N^{-1}$. Thus the impossibility of (A 5), and consequently of (A 4), is demonstrated if $r \leqslant \rho(p)$. This completes the proof of (A 1) under the second of the alternative conditions of theorem A.

To prove (A 2) under these conditions, we consider as before the completely continuous vector fields

$$F(u, t) = u - t\tilde{A}u \quad (0 \leqslant t \leqslant 1)$$

on S_R, which is taken to be some sufficiently large sphere; and we establish the homotopy of $I - \tilde{A}$ and I on S_R by showing that F cannot vanish for $0 < t \leqslant 1$. Again the proof is by contradiction: that is, we assume a vector u_0 to exist such that $u_0 = t\tilde{A}u_0$, which at once implies that $u_0 \in K$ and so

$$u_0 = tAu_0 \tag{A 12}$$

for some value of t satisfying $0 < t \leqslant 1$.

Now, a condition of theorem III is that the strong asymptotic derivative $A'(\infty)$ with respect to the cone has no positive eigenvector corresponding to an eigenvalue greater than or equal to unity, and this implies that a number $\beta > 0$ can be found such that

$$\|u_0 - tA'(\infty)\,u_0\| \geqslant \beta\|u_0\| \quad \text{if} \quad u_0 \in K,\ 0 \leqslant t \leqslant 1. \tag{A 13}$$

Also, by the definition of $A'(\infty)$ [see (2.5)], a finite number $\mathscr{R}$ can be found that is sufficiently large for $\|u_0\| = R \geqslant \mathscr{R}$ to imply

$$\|Au_0 - A'(\infty)\,u_0\| \leqslant \tfrac{1}{2}\beta\|u_0\| \quad \text{if} \quad u_0 \in K. \tag{A 14}$$

Combining (A 13) and (A 14) with the triangle inequality for norms, we obtain

$$\left.\begin{aligned} \|u_0 - tAu_0\| &\geqslant \|u_0 - tA'(\infty)\,u_0\| - t\|Au_0 - A'(\infty)\,u_0\| \\ &\geqslant \beta(1-\tfrac{1}{2}t)\,\|u_0\| \geqslant \tfrac{1}{2}\beta R \\ \text{if} \qquad & u_0 \in K, \quad 0 \leqslant t \leqslant 1, \quad R \geqslant \mathscr{R}. \end{aligned}\right\} \tag{A 15}$$

Thus an obvious contradiction of (A 12) is presented if we take $R \geqslant \mathscr{R}$, and hence we conclude that (A 2) holds. The proof of theorem A is now complete.

Theorem B can be proved similarly. Under the first of the alternative conditions, the properties (3.22) and (3.23) are deducible in exactly the same way as (3.21, A 2) and (3.20, A 1) under the first alternative for theorem A: that is, the arguments respecting S_r and S_R are simply interchanged. For the proof of theorem B under the second of the alternative conditions, the methods used above can be adapted without requiring any essentially new ideas.

Appendix 2. The variational property of conjugate flows

Here we examine a principle introduced in § 3.5, namely that conjugate flows in frictionless systems generally have the property of making an expression for flow force stationary. Examples of this property were considered in §§ 4 and 6.5, and various others can be given (e.g. axisymmetric vortex flows: see Benjamin 1962). Our purpose is to develop a general argument showing why the property arises.

By definition conjugate flows are x-independent, but the present issue is essentially connected with the problem of steady flows depending on x, i.e. the problem expressed in one possible form by (1.1). A model with an ample degree of generality is given by supposing that the hydrodynamical equations lead, under the hypothesis of steady flow, to a nonlinear elliptic equation of second order, and that this equation has a form derivable from a variational principle. The physical variable satisfying the equation is denoted by $\psi(x, y)$, which could be (but is not necessarily) a stream-function; and the independent variable y is taken to be a scalar, i.e. a single coordinate sufficient to describe the cross section of the flow. (It is a simple matter to extend the following argument to cases where two coordinates are needed to describe the cross-section.) The boundary conditions are taken, for example, to be that

$$\psi(x, 0) = 0, \quad \psi(x, 1) = \text{const.}, \tag{A 16}$$

and that ψ is periodic with respect to x on an interval $[0, l]$. We note that problems involving free boundaries can generally be put into a form with fixed boundary conditions by a suitable choice of variables (see Benjamin 1966, § 3.1): these boundary conditions may not be the same as (A 16), but with obvious modifications the following argument will still carry through. We also note that for problems in which conjugate flows are representable by finite-dimensional vectors (e.g. the problem of discretely stratified fluids), the present theory may be made directly applicable by considering them as extreme examples, given by taking a limit in which the primary fluid properties (e.g. density) become piecewise constant with discontinuities at certain interfaces (cf. Benjamin 1966, p. 263). Thus, in consideration of these various extensions of the argument, a very wide range of problems is covered in principle.

According to our assumptions there is a functional

$$\Pi = \int_0^l \int_0^1 \mathscr{I}(y, \psi, \psi_x, \psi_y)\, \mathrm{d}x\, \mathrm{d}y, \tag{A 17}$$

whose Euler–Lagrange equation,

$$\mathscr{E}(\psi) = \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{I}}{\partial \psi_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathscr{I}}{\partial \psi_y}\right) - \frac{\partial \mathscr{I}}{\partial \psi} = 0, \tag{A 18}$$

is the second-order differential equation in question. Note that $\mathscr{I}$—and hence the coefficients of (A 18)—cannot depend explicitly on x because of the periodicity condition or, what amounts to the same thing, because the prescribed basic properties of the flow system are uniform in the x-direction. A precise domain of definition for the functional Π need not concern us here, since the aim is not to answer questions of the existence of solutions. We assume that ψ is a 'classical' solution of (A 18) and the boundary conditions.

Defining

$$P = \mathscr{I} - \psi_x \frac{\partial \mathscr{I}}{\partial \psi_x}, \quad Q = \psi_x \frac{\partial \mathscr{I}}{\partial \psi_y}, \tag{A 19}$$

we find directly that

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = -\psi_x \mathscr{E}(\psi) = 0. \tag{A 20}$$

But $\partial P/\partial x = \partial Q/\partial y$ is the necessary and sufficient condition for $P\,\mathrm{d}y + Q\,\mathrm{d}x$ to be an exact differential, that is, for the line integral

$$\int (P\,\mathrm{d}y + Q\,\mathrm{d}x) \tag{A 21}$$

to be independent of the path between any two points in the flow domain.

So far we have considered the mathematical problem in abstract terms, but at this last step the essential connexion with a momentum principle may be appreciated. For it appears that the *only* physical principle expressed non-trivially by the invariance of a line integral like (A 21) is the one for momentum conservation respective to the x-direction (the conservation in steady flows of mass, energy flux and—in the case of swirling flows—angular momentum, for instance, are all expressible by a trivial form of such an integral, where the integrand is identically a total differential). If y is a Cartesian coordinate this principle is expressed by the invariance of

$$\int \{(p + \rho u^2)\,\mathrm{d}y - \rho uv\,\mathrm{d}x\}, \tag{A 22}$$

where u and v are the velocity components respective to x and y, p is pressure and ρ is density (cf. Benjamin 1966, p. 245). Thus $p + \rho u^2$ must be proportional to P, and $-\rho uv$ to Q, in the abstract statement of the problem. Admittedly this is a roundabout way of looking at the matter. In a specific application one would start rather with the physical principle, then express $P = p + \rho u^2$ and $Q = -\rho uv$ in terms of the single dependent variable ψ and its derivatives, and finally use $\partial P/\partial x = \partial Q/\partial y$ to obtain the governing equation for ψ, after which one might recognize a variational principle. But the present approach emphasizes the generality of the following conclusions.

Now, flow force is defined by

$$S = \int_0^1 (p + \rho u^2)\,\mathrm{d}y, \tag{A 23}$$

and from what has been said it follows that, except perhaps for a constant factor which would be immaterial to the gist of our argument, we have

$$S = \int_0^1 P(y, \psi, \psi_x, \psi_y)\,\mathrm{d}y, \tag{A 24}$$

where P is defined by the first of (A 19). Using (A 18) and the boundary conditions (A 16), we may readily confirm from (A 24) that $\mathrm{d}S/\mathrm{d}x = 0$, which is an obvious physical property of flow force. Note, however, that the fact $\mathrm{d}S/\mathrm{d}x = 0$ is not by itself sufficient to establish (A 18) from (A 24), since it leaves the alternatives that $\mathscr{E}(\psi) = 0$ or that $\mathscr{E}(\psi)$ and ψ_x are orthogonal as functions of y on [0, 1] (cf. Benjamin 1966, p. 246).

In the particular case of x-independent flows, i.e. $\psi = \psi(y)$, flow force is given by

$$S = \int_0^1 I(y, \psi, \psi_y)\,\mathrm{d}y, \tag{A 25}$$

where according to (A 24) and (A 19)

$$I(y, \psi, \psi_y) = P(y, \psi, 0, \psi_y) = \mathscr{I}(y, \psi, 0, \psi_y). \tag{A 26}$$

Hence (A 18) shows that the ordinary differential equation satisfied by $\psi(y)$ is

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\partial I}{\partial \psi_y}\right) - \frac{\partial I}{\partial \psi} = 0, \tag{A 27}$$

which is just the Euler–Lagrange equation for the functional (A 25). Thus it appears, as anticipated, that x-independent solutions $\psi(y)$ which represent conjugate flows are extremals of the expression for flow force as a functional of $\psi(y)$.

This conclusion is obviously unaffected if we put $\psi(x, y) = \Psi(y) + \phi(x, y)$, where $\Psi(y)$ is a known function representing a primary flow, and we consider Π and S as functionals of ϕ. The null solution $\phi \equiv 0$ is then an extremal of Π and S, as also is a non-trivial x-independent solution, $\phi = \phi(y)$, of (A 27). This idea was exemplified in the discussion at the end of § 6.5.

The following fact introduced into the general conjugate-flow theory (§ 3.5) was also illustrated in § 6.5. For any problem of the present type, suppose that the system comprising a differential equation and boundary conditions is recast in an operator form derivable from a variational principle posed in respect of the Hilbert space L_2. Then the functional in question is defined on a much wider class of functions than is meaningful for the original form of the problem. But in the case of a *solution*, which has differentiability properties not common to the whole class from which it is drawn, this functional generally represents the same physical quantity as the functional with narrower domain of definition whose Euler–Lagrange equation is the original differential equation. In § 6.5 this idea was demonstrated with regard to the variational principle for x-independent flows, and we may illustrate it as follows with regard to the more basic piece of theory outlined above.

Suppose that $\Gamma_1(\phi) = \Pi(\Psi + \phi) - \Pi(\Psi)$ has the form

$$\Gamma_1(\phi) = \int_D \left\{\tfrac{1}{2} r(y)\,\phi_x^2 + \tfrac{1}{2} s(y)\,\phi_y^2 - \int_0^\phi F(y, z)\,\mathrm{d}z\right\} \mathrm{d}x\,\mathrm{d}y, \tag{A 28}$$

where D denotes for short the same domain of integration as in (A 17). Also, $r(y)$ and $s(y)$ are positive functions on [0, 1], and $F(y, z)$ is a Lipschitz-continuous nonlinear function of both variables such that $F(y, 0) = 0$ and

$$F(y, z) \leqslant a(y) + b|z|, \tag{A 29}$$

where $a(y) \in L_2(0,1)$ and b is a positive constant. The Euler–Lagrange equation for Γ_1 is

$$\left.\begin{aligned} \mathscr{L}\phi &= F(y,\phi), \\ -\mathscr{L}\phi &= r\phi_{xx} + (s\phi_y)_y, \end{aligned}\right\} \tag{A 30}$$

with

and the given boundary conditions are such that this elliptic differential operator is self-adjoint on D. It is well known that the inverse, $\mathscr{B}$ say, of $\mathscr{L}$ is a linear integral operator whose kernel k is the Green function satisfying $\mathscr{L}k(x,y;\hat{x},\hat{y}) = \delta(x-\hat{x})\,.\,\delta(y-\hat{y})$ and the boundary conditions. Thus $\mathscr{L}(\mathscr{B}\phi) = \phi$, and $\mathscr{B}$ is also self-adjoint on D. An operator equation representing the hydrodynamical problem is therefore

$$\phi - \mathscr{B}\boldsymbol{F}\phi = \theta, \tag{A 31}$$

where $\boldsymbol{F}\phi$ stands for $F\{y,\phi(x,y)\}$, considered as a transformation in $L_2(D)$. In this space $\boldsymbol{F}$ is a continuous and bounded operator by virtue of the condition (A 29); and hence, since $\mathscr{B}$ is completely continuous (e.g. see K (b), p. 19), the nonlinear operator $\mathscr{B}\boldsymbol{F}$ appearing in (A 31) is completely continuous.

The left-hand side of (A 31) is not the gradient of any functional, but we may proceed by using the same device as in § 6.5. It is easily seen that $\mathscr{B}$ is a positive operator in the Hilbert-space sense (cf. second paragraph of § 6.5), and so there exists a self-adjoint completely continuous operator $\mathscr{B}^{\frac{1}{2}}$ definable as the principal square-root of $\mathscr{B}$, i.e. such that $\mathscr{B}^{\frac{1}{2}}(\mathscr{B}^{\frac{1}{2}}u) = \mathscr{B}u$ if $u \in L_2(D)$. Also, $\mathscr{B}^{\frac{1}{2}}u = \theta$ only if $u = \theta$. Hence, putting $\phi = \mathscr{B}^{\frac{1}{2}}\zeta$, we obtain from (A 31) the equivalent equation

$$\zeta - \mathscr{B}^{\frac{1}{2}}\boldsymbol{F}(\mathscr{B}^{\frac{1}{2}}\zeta) = \theta, \tag{A 32}$$

which is also expressible as

$$\operatorname{grad}\Gamma_2(\zeta) = \theta,$$

where

$$\Gamma_2(\zeta) = \int_D \left\{\tfrac{1}{2}\zeta^2 - \int_0^{\mathscr{B}^{\frac{1}{2}}\zeta} F(y,z)\,\mathrm{d}z\right\}\mathrm{d}x\,\mathrm{d}y. \tag{A 33}$$

Thus the existence of a solution of the original problem might be proved by establishing a non-zero stationary point of this functional in $L_2(D)$. [Note, incidentally, that the existence problem may also be approached through a study of the functional Γ_1 defined by (A 28). The type of functions considered in the classical calculus of variations is too restricted to be useful for this purpose; but the variational principle could helpfully be posed in respect of one of the so-called Sobolev spaces. This is the collection of functions which have generalized partial derivatives and satisfy the boundary conditions in a generalized sense, and whose norm is given by the square root of the positive integral (A 34) below (cf. Berger & Berger 1968, § 4.3). In this approach the foregoing restrictions on $F(y,z)$ can be relaxed. The Sobolev-space method was not used in this paper because it seemed rather far removed from other methods that were needed, but it offers advantages that may be valuable in further studies of the conjugate-flow problem and related wave problems.]

The present aim is simply to confirm that $\Gamma_1(\phi)$ and $\Gamma_2(\zeta)$ are equivalent if ζ is a solution of (A 32), so that $\phi = \mathscr{B}^{\frac{1}{2}}\zeta$ is a solution of (A 31). By virtue of a well-known property common to linear operators such as $\mathscr{B}$ (the inverses of elliptic differential operators), the result

$$\phi = \mathscr{B}\boldsymbol{F}\phi \in L_2(D)$$

implies that ϕ has finite first partial derivatives. Hence, by the first assumption made about $F(y,z)$ after it was introduced in (A 28), the function $\boldsymbol{F}\phi$ is Lipschitz continuous. And from this fact together with (A 31) it follows that ϕ has finite second partial derivatives, being therefore a classical solution of the original boundary-value problem. Thus we have for sure that

$$i = \int_D (r\phi_x^2 + s\phi_y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_D \phi\mathscr{L}\phi\,\mathrm{d}x\,\mathrm{d}y \tag{A 34}$$

is defined in both forms, where the second follows from the first after an integration by parts. Putting $\phi = \mathscr{B}^{\frac{1}{2}}\zeta$ and using the self-adjointness of $\mathscr{B}^{\frac{1}{2}}$, we now find that

$$i = \int_D (\mathscr{B}^{\frac{1}{2}}\zeta)\,\mathscr{L}(\mathscr{B}^{\frac{1}{2}}\zeta)\,\mathrm{d}x\,\mathrm{d}y = \int_D \zeta\mathscr{L}(\mathscr{B}\zeta)\,\mathrm{d}x\,\mathrm{d}y = \int_D \zeta^2\,\mathrm{d}x\,\mathrm{d}y. \qquad \text{(A 35)}$$

In the light of (A 34) and (A 35), the functionals $\Gamma_1(\phi)$ and $\Gamma_2(\zeta)$ are seen to be equivalent in the case of a solution, as expected.

As was shown earlier, the variational principle respecting the flow force of conjugate flows is obtained simply by ignoring x-dependence in the present variational principle (i.e. Γ/l reduces to the flow-force difference Λ). The present conclusion also confirms, therefore, that equivalent expressions for flow force are given by the stationary functionals in the original and in the compact-operator form of the conjugate-flow problem.

Finally, let us reconsider the example of continuously stratified fluids, for which the theory of conjugate flows was discussed in § 6. In terms of a pseudo-stream-function $\psi(x,y)$ such that

$$\psi_x = -\rho^{\frac{1}{2}}v, \quad \psi_y = \rho^{\frac{1}{2}}u,$$

we have

$$P = p+\rho u^2 = H(\psi)+\tfrac{1}{2}(\psi_y^2-\psi_x^2)-gy\rho(\psi),$$
$$Q = -\rho uv = \psi_x\psi_y,$$

where the total head H and density ρ can be considered as functions of ψ alone. Hence the equation $\partial P/\partial x = \partial Q/\partial y$ established by momentum considerations leads immediately to

$$\Delta\psi + gy\rho'(\psi) - H'(\psi) = 0 \qquad \text{(A 36)}$$

[cf. Benjamin 1966, equation (2.13); Yih 1965, p. 76, equation (10)]. If alternatively the actual stream-function is taken as the dependent variable, the same argument leads just as directly to a rather more complicated second-order nonlinear equation, commonly called Long's equation [Benjamin 1966, equation (2.8); Yih 1965, p. 76, equation (11)]. Again, if the height η of the streamlines in a primary x-independent flow is taken as a 'semi-Lagrangian' coordinate and their height in the actual flow, say $y(x,\eta)$ is taken as the dependent variable (see Benjamin 1966, § 3.1), this argument leads to the correct equation satisfied by $y(x,\eta)$. The equation is also obtainable from the variational principle $\delta\Pi = 0$, after the integrand (A 37) below has been re-expressed in the new variables and $\mathrm{d}x\,\mathrm{d}y$ has been replaced by $y_\eta\,\mathrm{d}\eta\,\mathrm{d}x$. The case of a free upper boundary, which is defined by $\eta = 1$ even though $y(x,1)$ varies, is then covered quite simply.

The functional for which (A 36) is the Euler–Lagrange equation is (A 17) with

$$\mathscr{I} = H(\psi)+\tfrac{1}{2}(\psi_x^2+\psi_y^2)-gy\rho(\psi) \qquad \text{(A 37)}$$

(cf. Long 1953, p. 48). It is curious that the second and third parts of this integrand represent the difference between the kinetic-energy and potential-energy densities, thus at first sight suggesting some direct connexion with Hamilton's principle: a tentative interpretation in this direction was proposed by Long. But the whole integrand represents the quantity $p+\rho(u^2+v^2)$ which seems to have no immediately physical significance except when $v = 0$, in which case its integral over y is flow force.

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